46rd IMO 2005

Problem 1. Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths.
Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

Problem 2. Let $a_1, a_2, \ldots$ be a sequence of integers with infinitely many positive and negative terms. Suppose that for every positive integer n the numbers $a_1, a_2, \ldots, a_n$ leave n different remainders upon division by n.
Prove that every integer occurs exactly once in the sequence $a_1, a_2, \ldots$.

Problem 3. Let x, y, z be three positive reals such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$

Problem 4. Determine all positive integers relatively prime to all the terms of the infinite sequence

$$a_n = 2^n + 3^n + 6^n - 1, \; n \geq 1.$$

Problem 5. Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and BC not parallel with DA. Let two variable points E and F lie of the sides BC and DA, respectively and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R.
Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.

Problem 6. In a mathematical competition, in which 6 problems were posed to the participants, every two of these problems were solved by more than $\frac{2}{5}$ of the contestants. Moreover, no contestant solved all the 6 problems. Show that there are at least 2 contestants who solved exactly 5 problems each.

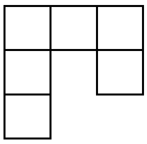
45rd IMO 2004

Problem 1. Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.

Problem 2. Find all polynomials f with real coefficients such that for all reals a,b,c such that $ab + bc + ca = 0$ we have the following relations

$$f(a - b) + f(b - c) + f(c - a) = 2f(a + b + c).$$

Problem 3. Define a "hook" to be a figure made up of six unit squares as shown below in the picture, or any of the figures obtained by applying rotations and reflections to this figure.



Determine all $m \times n$ rectangles that can be covered without gaps and without overlaps with hooks such that

- the rectangle is covered without gaps and without overlaps
- no part of a hook covers area outside the rectagle.

Problem 4. Let $n \geq 3$ be an integer. Let $t_1, t_2, ..., t_n$ be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + ... + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + ... + \frac{1}{t_n}\right).$$

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with
$1 \leq i < j < k \leq n$.

Problem 5. In a convex quadrilateral $ABCD$ the diagonal BD does not bisect the angles ABC and CDA. The point P lies inside $ABCD$ and satisfies

$$\angle PBC = \angle DBA \text{ and } \angle PDC = \angle BDA.$$

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

Problem 6. We call a positive integer *alternating* if every two consecutive digits in its decimal representation are of different parity.
Find all positive integers n such that n has a multiple which is alternating.

44th IMO 2003

Problem 1. S is the set $\{1, 2, 3, \ldots, 1000000\}$. Show that for any subset A of S with 101 elements we can find 100 distinct elements x_i of S, such that the sets $\{a + x_i | a \in A\}$ are all pairwise disjoint.

Problem 2. Find all pairs (m, n) of positive integers such that $\frac{m^2}{2mn^2 - n^3 + 1}$ is a positive integer.

Problem 3. A convex hexagon has the property that for any pair of opposite sides the distance between their midpoints is $\sqrt{3}/2$ times the sum of their lengths Show that all the hexagon's angles are equal.

Problem 4. $ABCD$ is cyclic. The feet of the perpendicular from D to the lines AB, BC, CA are P, Q, R respectively. Show that the angle bisectors of ABC and CDA meet on the line AC iff $RP = RQ$.

Problem 5. Given $n > 2$ and reals $x_1 \leq x_2 \leq \cdots \leq x_n$, show that $(\sum_{i,j} |x_i - x_j|)^2 \leq \frac{2}{3}(n^2 - 1) \sum_{i,j}(x_i - x_j)^2$. Show that we have equality iff the sequence is an arithmetic progression.

Problem 6. Show that for each prime p, there exists a prime q such that $n^p - p$ is not divisible by q for any positive integer n.

43rd IMO 2002

Problem 1. S is the set of all (h,k) with h,k non-negative integers such that $h+k<n$. Each element of S is colored red or blue, so that if (h,k) is red and $h'\leq h, k'\leq k$, then (h',k') is also red. A type 1 subset of S has n blue elements with different first member and a type 2 subset of S has n blue elements with different second member. Show that there are the same number of type 1 and type 2 subsets.

Problem 2. BC is a diameter of a circle center O. A is any point on the circle with $\angle AOC>60^o$. EF is the chord which is the perpendicular bisector of AO. D is the midpoint of the minor arc AB. The line through O parallel to AD meets AC at J. Show that J is the incenter of triangle CEF.

Problem 3. Find all pairs of integers $m>2, n>2$ such that there are infinitely many positive integers k for which k^n+k^2-1 divides k^m+k-1.

Problem 4. The positive divisors of the integer $n>1$ are $d_1<d_2<\ldots<d_k$, so that $d_1=1, d_k=n$. Let $d=d_1d_2+d_2d_3+\cdots+d_{k-1}d_k$. Show that $d<n^2$ and find all n for which d divides n^2.

Problem 5. Find all real-valued functions on the reals such that $(f(x)+f(y))((f(u)+f(v))=f(xu-yv)+f(xv+yu)$ for all x,y,u,v.

Problem 6. $n>2$ circles of radius 1 are drawn in the plane so that no line meets more than two of the circles. Their centers are $O_1,O_2,\cdots,O_n$. Show that $\sum_{i<j}1/O_iO_j\leq(n-1)\pi/4$.

42nd International Mathematical Olympiad

Washington, DC, United States of America
July 8–9, 2001

Problems

Each problem is worth seven points.

Problem 1

Let ABC be an acute-angled triangle with circumcentre O. Let P on BC be the foot of the altitude from A.

Suppose that $\angle BCA \geq \angle ABC + 30^\circ$.

Prove that $\angle CAB + \angle COP < 90^\circ$.

Problem 2

Prove that

$$\frac{a}{\sqrt{a^2 + 8\,b\,c}} + \frac{b}{\sqrt{b^2 + 8\,c\,a}} + \frac{c}{\sqrt{c^2 + 8\,a\,b}} \geq 1$$

for all positive real numbers a, b and c.

Problem 3

Twenty-one girls and twenty-one boys took part in a mathematical contest.

- Each contestant solved at most six problems.
- For each girl and each boy, at least one problem was solved by both of them.

Prove that there was a problem that was solved by at least three girls and at least three boys.

Problem 4

Let n be an odd integer greater than 1, and let $k_1, k_2, \ldots, k_n$ be given integers. For each of the $n!$ permutations $a = (a_1, a_2, \ldots, a_n)$ of $1, 2, \ldots, n$, let

$$S(a) = \sum_{i=1}^{n} k_i\, a_i.$$

Prove that there are two permutations b and c, $b \neq c$, such that $n!$ is a divisor of $S(b) - S(c)$.

Problem 5

In a triangle ABC, let AP bisect $\angle BAC$, with P on BC, and let BQ bisect $\angle ABC$, with Q on CA.

It is known that $\angle BAC = 60^\circ$ and that $AB + BP = AQ + QB$.

What are the possible angles of triangle ABC?

Problem 6

Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that

$$a\,c + b\,d = (b + d + a - c)\,(b + d - a + c).$$

Prove that $a\,b + c\,d$ is not prime.

41st IMO 2000

Problem 1. AB is tangent to the circles $CAMN$ and $NMBD$. M lies between C and D on the line CD, and CD is parallel to AB. The chords NA and CM meet at P; the chords NB and MD meet at Q. The rays CA and DB meet at E. Prove that $PE = QE$.

Problem 2. A, B, C are positive reals with product 1. Prove that $(A - 1 + \frac{1}{B})(B - 1 + \frac{1}{C})(C - 1 + \frac{1}{A}) \leq 1$.

Problem 3. k is a positive real. N is an integer greater than 1. N points are placed on a line, not all coincident. A *move* is carried out as follows. Pick any two points A and B which are not coincident. Suppose that A lies to the right of B. Replace B by another point B' to the right of A such that $AB' = kBA$. For what values of k can we move the points arbitrarily far to the right by repeated moves?

Problem 4. 100 cards are numbered 1 to 100 (each card different) and placed in 3 boxes (at least one card in each box). How many ways can this be done so that if two boxes are selected and a card is taken from each, then the knowledge of their sum alone is always sufficient to identify the third box?

Problem 5. Can we find N divisible by just 2000 different primes, so that N divides $2^N + 1$? [N may be divisible by a prime power.]

Problem 6. $A_1A_2A_3$ is an acute-angled triangle. The foot of the altitude from A_i is K_i and the incircle touches the side opposite A_i at L_i. The line K_1K_2 is reflected in the line L_1L_2. Similarly, the line K_2K_3 is reflected in L_2L_3 and K_3K_1 is reflected in L_3L_1. Show that the three new lines form a triangle with vertices on the incircle.

40^{th} International Mathematical Olympiad

Bucharest

Day I

July 16, 1999

1. Determine all finite sets S of at least three points in the plane which satisfy the following condition:

   for any two distinct points A and B in S, the perpendicular bisector of the line segment AB is an axis of symmetry for S.

2. Let n be a fixed integer, with $n \geq 2$.

   (a) Determine the least constant C such that the inequality

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{1 \leq i \leq n} x_i \right)^4$$

   holds for all real numbers $x_1, \cdots, x_n \geq 0$.

   (b) For this constant C, determine when equality holds.

3. Consider an $n \times n$ square board, where n is a fixed even positive integer. The board is divided into n^2 unit squares. We say that two different squares on the board are adjacent if they have a common side.

   N unit squares on the board are marked in such a way that every square (marked or unmarked) on the board is adjacent to at least one marked square.

   Determine the smallest possible value of N.

40^{th} International Mathematical Olympiad

Bucharest

Day II

July 17, 1999

4. Determine all pairs (n, p) of positive integers such that

   p is a prime,
   n not exceeded $2p$, and
   $(p-1)^n + 1$ is divisible by n^{p-1}.

5. Two circles G_1 and G_2 are contained inside the circle G, and are tangent to G at the distinct points M and N, respectively. G_1 passes through the center of G_2. The line passing through the two points of intersection of G_1 and G_2 meets G at A and B. The lines MA and MB meet G_1 at C and D, respectively.

   Prove that CD is tangent to G_2.

6. Determine all functions $f : \mathbf{R} \longrightarrow \mathbf{R}$ such that

   $$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

   for all real numbers x, y.

39^{th} International Mathematical Olympiad

Taipei, Taiwan

Day I

July 15, 1998

1. In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose that the point P, where the perpendicular bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles ABP and CDP have equal areas.

2. In a competition, there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that $k/a \geq (b-1)/(2b)$.

3. For any positive integer n, let $d(n)$ denote the number of positive divisors of n (including 1 and n itself). Determine all positive integers k such that $d(n^2)/d(n) = k$ for some n.

39^{th} International Mathematical Olympiad

Taipei, Taiwai

Day II

July 16, 1998

4. Determine all pairs (a, b) of positive integers such that $ab^2 + b + 7$ divides $a^2b + a + b$.

5. Let I be the incenter of triangle ABC. Let the incircle of ABC touch the sides BC, CA, and AB at K, L, and M, respectively. The line through B parallel to MK meets the lines LM and LK at R and S, respectively. Prove that angle RIS is acute.

6. Consider all functions f from the set N of all positive integers into itself satisfying $f(t^2f(s)) = s(f(t))^2$ for all s and t in N. Determine the least possible value of $f(1998)$.

38^{th} International Mathematical Olympiad

Mar del Plata, Argentina

Day I

July 24, 1997

1. In the plane the points with integer coordinates are the vertices of unit squares. The squares are colored alternately black and white (as on a chessboard).

   For any pair of positive integers m and n, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n, lie along edges of the squares.

   Let S_1 be the total area of the black part of the triangle and S_2 be the total area of the white part. Let

$$f(m, n) = |S_1 - S_2|.$$

   (a) Calculate $f(m, n)$ for all positive integers m and n which are either both even or both odd.

   (b) Prove that $f(m, n) \leq \frac{1}{2} \max\{m, n\}$ for all m and n.

   (c) Show that there is no constant C such that $f(m, n) < C$ for all m and n.

2. The angle at A is the smallest angle of triangle ABC. The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respectively. The lines BV and CW meet at T. Show that

$$AU = TB + TC.$$

3. Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying the conditions

$$|x_1 + x_2 + \cdots + x_n| = 1$$

   and

$$|x_i| \leq \frac{n+1}{2} \qquad i = 1, 2, \ldots, n.$$

   Show that there exists a permutation $y_1, y_2, \ldots, y_n$ of $x_1, x_2, \ldots, x_n$ such that

$$|y_1 + 2y_2 + \cdots + ny_n| \leq \frac{n+1}{2}.$$

38^{th} International Mathematical Olympiad

Mar del Plata, Argentina

Day II

July 25, 1997

4. An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a *silver* matrix if, for each $i = 1, 2, \ldots, n$, the ith row and the ith column together contain all elements of S. Show that

   (a) there is no silver matrix for $n = 1997$;

   (b) silver matrices exist for infinitely many values of n.

5. Find all pairs (a, b) of integers $a, b \geq 1$ that satisfy the equation

$$a^{b^2} = b^a.$$

6. For each positive integer n , let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways:

$$4; 2+2; 2+1+1; 1+1+1+1.$$

   Prove that, for any integer $n \geq 3$,

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}.$$

37^{th} International Mathematical Olympiad

Mumbai, India

Day I 9 a.m. - 1:30 p.m.

July 10, 1996

1. We are given a positive integer r and a rectangular board $ABCD$ with dimensions $|AB| = 20, |BC| = 12$. The rectangle is divided into a grid of 20×12 unit squares. The following moves are permitted on the board: one can move from one square to another only if the distance between the centers of the two squares is $\sqrt{r}$. The task is to find a sequence of moves leading from the square with A as a vertex to the square with B as a vertex.

   (a) Show that the task cannot be done if r is divisible by 2 or 3.

   (b) Prove that the task is possible when $r = 73$.

   (c) Can the task be done when $r = 97$?

2. Let P be a point inside triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

   Let D, E be the incenters of triangles APB, APC, respectively. Show that AP, BD, CE meet at a point.

3. Let S denote the set of nonnegative integers. Find all functions f from S to itself such that

$$f(m + f(n)) = f(f(m)) + f(n) \qquad \forall m, n \in S.$$

37^{th} International Mathematical Olympiad

Mumbai, India

Day II 9 a.m. - 1:30 p.m.

July 11, 1996

1. The positive integers a and b are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

2. Let $ABCDEF$ be a convex hexagon such that AB is parallel to DE, BC is parallel to EF, and CD is parallel to FA. Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF, respectively, and let P denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{P}{2}.$$

3. Let p, q, n be three positive integers with $p+q < n$. Let $(x_0, x_1, \ldots, x_n)$ be an $(n+1)$-tuple of integers satisfying the following conditions:

   (a) $x_0 = x_n = 0$.

   (b) For each i with $1 \leq i \leq n$, either $x_i - x_{i-1} = p$ or $x_i - x_{i-1} = -q$.

   Show that there exist indices $i < j$ with $(i, j) \neq (0, n)$, such that $x_i = x_j$.

36^{th} International Mathematical Olympiad

First Day - Toronto - July 19, 1995
Time Limit: $4\frac{1}{2}$ hours

1. Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

2. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

3. Determine all integers $n > 3$ for which there exist n points $A_1, \ldots, A_n$ in the plane, no three collinear, and real numbers $r_1, \ldots, r_n$ such that for $1 \leq i < j < k \leq n$, the area of $\triangle A_i A_j A_k$ is $r_i + r_j + r_k$.

36^{th} International Mathematical Olympiad

Second Day - Toronto - July 20, 1995
Time Limit: $4\frac{1}{2}$ hours

1. Find the maximum value of x_0 for which there exists a sequence $x_0, x_1 \ldots, x_{1995}$ of positive reals with $x_0 = x_{1995}$, such that for $i = 1, \ldots, 1995$,

$$x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i}.$$

2. Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$ and $DE = EF = FA$, such that $\angle BCD = \angle EFA = \pi/3$. Suppose G and H are points in the interior of the hexagon such that $\angle AGB = \angle DHE = 2\pi/3$. Prove that $AG + GB + GH + DH + HE \geq CF$.

3. Let p be an odd prime number. How many p-element subsets A of $\{1, 2, \ldots\ 2p\}$ are there, the sum of whose elements is divisible by p?

The 35th International Mathematical Olympiad (July 13-14, 1994, Hong Kong)

1. Let m and n be positive integers. Let $a_1, a_2, \ldots, a_m$ be distinct elements of $\{1, 2, \ldots, n\}$ such that whenever $a_i + a_j \le n$ for some i, j, $1 \le i \le j \le m$, there exists k, $1 \le k \le m$, with $a_i + a_j = a_k$. Prove that

$$\frac{a_1 + a_2 + \cdots + a_m}{m} \ge \frac{n+1}{2}.$$

2. ABC is an isosceles triangle with $AB = AC$. Suppose that

1. M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB;
2. Q is an arbitrary point on the segment BC different from B and C;
3. E lies on the line AB and F lies on the line AC such that E, Q, F are distinct and collinear.

Prove that OQ is perpendicular to EF if and only if $QE = QF$.

3. For any positive integer k, let $f(k)$ be the number of elements in the set $\{k+1, k+2, \ldots, 2k\}$ whose base 2 representation has precisely three 1s.

- (a) Prove that, for each positive integer m, there exists at least one positive integer k such that $f(k) = m$.
- (b) Determine all positive integers m for which there exists exactly one k with $f(k) = m$.

4. Determine all ordered pairs (m, n) of positive integers such that

$$\frac{n^3 + 1}{mn - 1}$$

is an integer.

5. Let S be the set of real numbers strictly greater than -1. Find all functions $f : S \to S$ satisfying the two conditions:

1. $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all x and y in S;
2. $\frac{f(x)}{x}$ is strictly increasing on each of the intervals $-1 < x < 0$ and $0 < x$.

6. Show that there exists a set A of positive integers with the following property: For any infinite set S of primes there exist two positive integers $m \in A$ and $n \notin A$ each of which is a product of k distinct elements of S for some $k \ge 2$.

34nd International Mathematical Olympiad

First Day —July 18, 1993
Time Limit: $4\frac{1}{2}$ hours

1. Let $f(x) = x^n + 5x^{n-1} + 3$, where $n > 1$ is an integer. Prove that $f(x)$ cannot be expressed as the product of two nonconstant polynomials with integer coefficients.

2. Let D be a point inside acute triangle ABC such that $\angle ADB = \angle ACB + \pi/2$ and $AC \cdot BD = AD \cdot BC$.

   (a) Calculate the ratio $(AB \cdot CD)/(AC \cdot BD)$.

   (b) Prove that the tangents at C to the circumcircles of $\triangle ACD$ and $\triangle BCD$ are perpendicular.

3. On an infinite chessboard, a game is played as follows. At the start, n^2 pieces are arranged on the chessboard in an n by n block of adjoining squares, one piece in each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square to an unoccupied square immediately beyond. The piece which has been jumped over is removed.

   Find those values of n for which the game can end with only one piece remaining on the board.

Second Day —July 19, 1993
Time Limit: $4\frac{1}{2}$ hours

1. For three points P, Q, R in the plane, we define $m(PQR)$ as the minimum length of the three altitudes of $\triangle PQR$. (If the points are collinear, we set $m(PQR) = 0$.)

   Prove that for points A, B, C, X in the plane,

   $$m(ABC) \leq m(ABX) + m(AXC) + m(XBC).$$

2. Does there exist a function $f : \mathbf{N} \to \mathbf{N}$ such that $f(1) = 2$, $f(f(n)) = f(n) + n$ for all $n \in \mathbf{N}$, and $f(n) < f(n+1)$ for all $n \in \mathbf{N}$?

3. There are n lamps $L_0, \ldots, L_{n-1}$ in a circle ($n > 1$), where we denote $L_{n+k} = L_k$. (A lamp at all times is either on or off.) Perform steps $s_0, s_1, \ldots$ as follows: at step s_i, if L_{i-1} is lit, switch L_i from on to off or vice versa, otherwise do nothing. Initially all lamps are on. Show that:

   (a) There is a positive integer $M(n)$ such that after $M(n)$ steps all the lamps are on again;

   (b) If $n = 2^k$, we can take $M(n) = n^2 - 1$;

   (c) If $n = 2^k + 1$, we can take $M(n) = n^2 - n + 1$.

33^{rd} International Mathematical Olympiad

First Day - Moscow - July 15, 1992
Time Limit: $4\frac{1}{2}$ hours

1. Find all integers a, b, c with $1 < a < b < c$ such that

$$(a-1)(b-1)(c-1) \quad \text{is a divisor of } abc-1.$$

2. Let $\mathbf{R}$ denote the set of all real numbers. Find all functions $f : \mathbf{R} \to \mathbf{R}$ such that

$$f\left(x^2 + f(y)\right) = y + (f(x))^2 \quad \text{for all } x, y \in R.$$

3. Consider nine points in space, no four of which are coplanar. Each pair of points is joined by an edge (that is, a line segment) and each edge is either colored blue or red or left uncolored. Find the smallest value of n such that whenever exactly n edges are colored, the set of colored edges necessarily contains a triangle all of whose edges have the same color.

33^{rd} International Mathematical Olympiad

Second Day - Moscow - July 15, 1992
Time Limit: $4\frac{1}{2}$ hours

1. In the plane let C be a circle, L a line tangent to the circle C, and M a point on L. Find the locus of all points P with the following property: there exists two points Q, R on L such that M is the midpoint of QR and C is the inscribed circle of triangle PQR.

2. Let S be a finite set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane, respectively. Prove that

$$|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|,$$

where $|A|$ denotes the number of elements in the finite set $|A|$. (Note: The orthogonal projection of a point onto a plane is the foot of the perpendicular from that point to the plane.)

3. For each positive integer n, $S(n)$ is defined to be the greatest integer such that, for every positive integer $k \leq S(n)$, n^2 can be written as the sum of k positive squares.

   (a) Prove that $S(n) \leq n^2 - 14$ for each $n \geq 4$.

   (b) Find an integer n such that $S(n) = n^2 - 14$.

   (c) Prove that there are infintely many integers n such that $S(n) = n^2 - 14$.

32nd International Mathematical Olympiad

First Day —July 17, 1991
Time Limit: $4\frac{1}{2}$ hours

1. Given a triangle ABC, let I be the center of its inscribed circle. The internal bisectors of the angles A, B, C meet the opposite sides in A', B', C' respectively. Prove that

$$\frac{1}{4} < \frac{AI \cdot BI \cdot CI}{AA' \cdot BB' \cdot CC'} \leq \frac{8}{27}.$$

2. Let $n > 6$ be an integer and $a_1, a_2, \ldots, a_k$ be all the natural numbers less than n and relatively prime to n. If

$$a_2 - a_1 = a_3 - a_2 = \cdots = a_k - a_{k-1} > 0,$$

prove that n must be either a prime number or a power of 2.

3. Let $S = \{1, 2, 3, \ldots, 280\}$. Find the smallest integer n such that each n-element subset of S contains five numbers which are pairwise relatively prime.

Second Day —July 18, 1991
Time Limit: $4\frac{1}{2}$ hours

1. Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2, \ldots, k$ in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is equal to 1.

   [A *graph* consists of a set of points, called *vertices*, together with a set of *edges* joining certain pairs of distinct vertices. Each pair of vertices u, v belongs to at most one edge. The graph G is *connected* if for each pair of distinct vertices x, y there is some sequence of vertices $x = v_0, v_1, v_2, \ldots, v_m = y$ such that each pair v_i, v_{i+1} $(0 \leq i < m)$ is joined by an edge of G.]

2. Let ABC be a triangle and P an interior point of ABC. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to $30°$.

3. An infinite sequence $x_0, x_1, x_2, \ldots$ of real numbers is said to be *bounded* if there is a constant C such that $|x_i| \leq C$ for every $i \geq 0$.

   Given any real number $a > 1$, construct a bounded infinite sequence $x_0, x_1, x_2, \ldots$ such that

$$|x_i - x_j||i - j|^a \geq 1$$

   for every pair of distinct nonnegative integers i, j.

31^{st} International Mathematical Olympiad

Beijing, China

Day I

July 12, 1990

1. Chords AB and CD of a circle intersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangent line at E to the circle through D, E, and M intersects the lines BC and AC at F and G, respectively. If
$$\frac{AM}{AB} = t,$$
find
$$\frac{EG}{EF}$$
in terms of t.

2. Let $n \geq 3$ and consider a set E of $2n - 1$ distinct points on a circle. Suppose that exactly k of these points are to be colored black. Such a coloring is "good" if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly n points from E. Find the smallest value of k so that every such coloring of k points of E is good.

3. Determine all integers $n > 1$ such that
$$\frac{2^n + 1}{n^2}$$
is an integer.

31^{st} International Mathematical Olympiad

Beijing, China

Day II

July 13, 1990

4. Let $\mathbb{Q}^+$ be the set of positive rational numbers. Construct a function $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that
$$f(xf(y)) = \frac{f(x)}{y}$$
for all x, y in $\mathbb{Q}^+$.

5. Given an initial integer $n_0 > 1$, two players, $\mathcal{A}$ and $\mathcal{B}$, choose integers n_1, n_2, n_3, ... alternately according to the following rules:

   Knowing n_{2k}, $\mathcal{A}$ chooses any integer n_{2k+1} such that
$$n_{2k} \leq n_{2k+1} \leq n_{2k}^2.$$
   Knowing n_{2k+1}, $\mathcal{B}$ chooses any integer n_{2k+2} such that
$$\frac{n_{2k+1}}{n_{2k+2}}$$
   is a prime raised to a positive integer power.

   Player $\mathcal{A}$ wins the game by choosing the number 1990; player $\mathcal{B}$ wins by choosing the number 1. For which n_0 does:

   (a) $\mathcal{A}$ have a winning strategy?

   (b) $\mathcal{B}$ have a winning strategy?

   (c) Neither player have a winning strategy?

6. Prove that there exists a convex 1990-gon with the following two properties:

   (a) All angles are equal.

   (b) The lengths of the 1990 sides are the numbers 1^2, 2^2, 3^2, ..., 1990^2 in some order.

30^{th} International Mathematical Olympiad

Braunschweig, Germany

Day I

1. Prove that the set $\{1, 2, \ldots, 1989\}$ can be expressed as the disjoint union of subsets A_i ($i = 1, 2, \ldots, 117$) such that:

   (i) Each A_i contains 17 elements;

   (ii) The sum of all the elements in each A_i is the same.

2. In an acute-angled triangle ABC the internal bisector of angle A meets the circumcircle of the triangle again at A_1. Points B_1 and C_1 are defined similarly. Let A_0 be the point of intersection of the line AA_1 with the external bisectors of angles B and C. Points B_0 and C_0 are defined similarly. Prove that:

   (i) The area of the triangle $A_0B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$.

   (ii) The area of the triangle $A_0B_0C_0$ is at least four times the area of the triangle ABC.

3. Let n and k be positive integers and let S be a set of n points in the plane such that

   (i) No three points of S are collinear, and

   (ii) For any point P of S there are at least k points of S equidistant from P.

   Prove that:

$$k < \frac{1}{2} + \sqrt{2n}.$$

30^{th} International Mathematical Olympiad

Braunschweig, Germany

Day II

4. Let $ABCD$ be a convex quadrilateral such that the sides AB, AD, BC satisfy $AB = AD + BC$. There exists a point P inside the quadrilateral at a distance h from the line CD such that $AP = h + AD$ and $BP = h + BC$. Show that:

$$\frac{1}{\sqrt{h}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}.$$

5. Prove that for each positive integer n there exist n consecutive positive integers none of which is an integral power of a prime number.

6. A permutation $(x_1, x_2, \ldots, x_m)$ of the set $\{1, 2, \ldots, 2n\}$, where n is a positive integer, is said to have property P if $|x_i - x_{i+1}| = n$ for at least one i in $\{1, 2, \ldots, 2n-1\}$. Show that, for each n, there are more permutations with property P than without.

28^{th} International Mathematical Olympiad

Havana, Cuba

Day I

July 10, 1987

1. Let $p_n(k)$ be the number of permutations of the set $\{1, \ldots, n\}$, $n \geq 1$, which have exactly k fixed points. Prove that

$$\sum_{k=0}^{n} k \cdot p_n(k) = n!.$$

(Remark: A permutation f of a set S is a one-to-one mapping of S onto itself. An element i in S is called a fixed point of the permutation f if $f(i) = i$.)

2. In an acute-angled triangle ABC the interior bisector of the angle A intersects BC at L and intersects the circumcircle of ABC again at N. From point L perpendiculars are drawn to AB and AC, the feet of these perpendiculars being K and M respectively. Prove that the quadrilateral $AKNM$ and the triangle ABC have equal areas.

3. Let $x_1, x_2, \ldots, x_n$ be real numbers satisfying $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$. Prove that for every integer $k \geq 2$ there are integers $a_1, a_2, \ldots, a_n$, not all 0, such that $|a_i| \leq k - 1$ for all i and

$$|a_1x_1 + a_1x_2 + \cdots + a_nx_n| \leq \frac{(k-1)\sqrt{n}}{k^n - 1}.$$

28^{th} International Mathematical Olympiad

Havana, Cuba

Day II

July 11, 1987

4. Prove that there is no function f from the set of non-negative integers into itself such that $f(f(n)) = n + 1987$ for every n.

5. Let n be an integer greater than or equal to 3. Prove that there is a set of n points in the plane such that the distance between any two points is irrational and each set of three points determines a non-degenerate triangle with rational area.

6. Let n be an integer greater than or equal to 2. Prove that if $k^2 + k + n$ is prime for all integers k such that $0 \le k \le \sqrt{n/3}$, then $k^2 + k + n$ is prime for all integers k such that $0 \le k \le n - 2$.

27^{th} International Mathematical Olympiad

Warsaw, Poland

Day I

July 9, 1986

1. Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a, b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.

2. A triangle $A_1A_2A_3$ and a point P_0 are given in the plane. We define $A_s = A_{s-3}$ for all $s \geq 4$. We construct a set of points P_1, P_2, P_3, $\ldots$, such that P_{k+1} is the image of P_k under a rotation with center A_{k+1} through angle $120°$ clockwise (for $k = 0, 1, 2, \ldots$). Prove that if $P_{1986} = P_0$, then the triangle $A_1A_2A_3$ is equilateral.

3. To each vertex of a regular pentagon an integer is assigned in such a way that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively and $y < 0$ then the following operation is allowed: the numbers x, y, z are replaced by $x + y$, $-y$, $z + y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to and end after a finite number of steps.

27^{th} International Mathematical Olympiad

Warsaw, Poland

Day II

July 10, 1986

4. Let A, B be adjacent vertices of a regular n-gon ($n \geq 5$) in the plane having center at O. A triangle XYZ, which is congruent to and initially conincides with OAB, moves in the plane in such a way that Y and Z each trace out the whole boundary of the polygon, X remaining inside the polygon. Find the locus of X.

5. Find all functions f, defined on the non-negative real numbers and taking non-negative real values, such that:

   (i) $f(xf(y))f(y) = f(x+y)$ for all x, $y \geq 0$,

   (ii) $f(2) = 0$,

   (iii) $f(x) \neq 0$ for $0 \leq x < 2$.

6. One is given a finite set of points in the plane, each point having integer coordinates. Is it always possible to color some of the points in the set red and the remaining points white in such a way that for any straight line L parallel to either one of the coordinate axes the difference (in absolute value) between the numbers of white point and red points on L is not greater than 1?

Twenty-sixth International Olympiad, 1985

1985/1. A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

1985/2. Let n and k be given relatively prime natural numbers, $k < n$. Each number in the set $M = \{1, 2, ..., n-1\}$ is colored either blue or white. It is given that

(i) for each $i \in M$, both i and $n - i$ have the same color;

(ii) for each $i \in M, i \neq k$, both i and $|i - k|$ have the same color. Prove that all numbers in M must have the same color.

1985/3. For any polynomial $P(x) = a_0 + a_1x + \cdots + a_kx^k$ with integer coefficients, the number of coefficients which are odd is denoted by $w(P)$. For $i = 0, 1, ...,$ let $Q_i(x) = (1 + x)^i$. Prove that if $i_1i_2, ..., i_n$ are integers such that $0 \leq i_1 < i_2 < \cdots < i_n$, then

$$w(Q_{i_1} + Q_{i_2}, + + Q_{i_n}) \geq w(Q_{i_1}).$$

1985/4. Given a set M of 1985 distinct positive integers, none of which has a prime divisor greater than 26. Prove that M contains at least one subset of four distinct elements whose product is the fourth power of an integer.

1985/5. A circle with center O passes through the vertices A and C of triangle ABC and intersects the segments AB and BC again at distinct points K and N, respectively. The circumscribed circles of the triangles ABC and EBN intersect at exactly two distinct points B and M. Prove that angle OMB is a right angle.

1985/6. For every real number x_1, construct the sequence $x_1, x_2, ...$ by setting

$$x_{n+1} = x_n \left(x_n + \frac{1}{n} \right) \text{ for each } n \geq 1.$$

Prove that there exists exactly one value of x_1 for which

$$0 < x_n < x_{n+1} < 1$$

for every n.

Twenty-fifth International Olympiad, 1984

1984/1. Prove that $0 \leq yz + zx + xy - 2xyz \leq 7/27$, where x, y and z are non-negative real numbers for which $x + y + z = 1$.

1984/2. Find one pair of positive integers a and b such that:
(i) $ab(a + b)$ is not divisible by 7;
(ii) $(a + b)^7 - a^7 - b^7$ is divisible by 7^7 .
Justify your answer.

1984/3. In the plane two different points O and A are given. For each point X of the plane, other than O, denote by $a(X)$ the measure of the angle between OA and OX in radians, counterclockwise from $OA (0 \leq a(X) < 2\pi)$. Let $C(X)$ be the circle with center O and radius of length $OX + a(X)/OX$. Each point of the plane is colored by one of a finite number of colors. Prove that there exists a point Y for which $a(Y) > 0$ such that its color appears on the circumference of the circle $C(Y)$.

1984/4. Let $ABCD$ be a convex quadrilateral such that the line CD is a tangent to the circle on AB as diameter. Prove that the line AB is a tangent to the circle on CD as diameter if and only if the lines BC and AD are parallel.

1984/5. Let d be the sum of the lengths of all the diagonals of a plane convex polygon with n vertices $(n > 3)$, and let p be its perimeter. Prove that

$$n - 3 < \frac{2d}{p} < \left[\frac{n}{2}\right]\left[\frac{n+1}{2}\right] - 2,$$

where $[x]$ denotes the greatest integer not exceeding x.

1984/6. Let a, b, c and d be odd integers such that $0 < a < b < c < d$ and $ad = bc$. Prove that if $a + d = 2^k$ and $b + c = 2^m$ for some integers k and m, then $a = 1$.

Twenty-fourth International Olympiad, 1983

1983/1. Find all functions f defined on the set of positive real numbers which take positive real values and satisfy the conditions:
(i) $f(xf(y)) = yf(x)$ for all positive x, y;
(ii) $f(x) \to 0$ as $x \to \infty$.

1983/2. Let A be one of the two distinct points of intersection of two unequal coplanar circles C_1 and C_2 with centers O_1 and O_2, respectively. One of the common tangents to the circles touches C_1 at P_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1,and M_2 be the midpoint of P_2Q_2. Prove that $\angle O_1AO_2 = \angle M_1AM_2$.

1983/3. Let a, b and c be positive integers, no two of which have a common divisor greater than 1. Show that $2abc - ab - bc - ca$ is the largest integer which cannot be expressed in the form $xbc + yca + zab$,where x, y and z are non-negative integers.

1983/4. Let ABC be an equilateral triangle and $\mathcal{E}$ the set of all points contained in the three segments AB, BC and CA (including A, B and C). Determine whether, for every partition of $\mathcal{E}$ into two disjoint subsets, at least one of the two subsets contains the vertices of a right-angled triangle. Justify your answer.

1983/5. Is it possible to choose 1983 distinct positive integers, all less than or equal to 10^5, no three of which are consecutive terms of an arithmetic progression? Justify your answer.

1983/6. Let a, b and c be the lengths of the sides of a triangle. Prove that

$$a^2b(a - b) + b^2c(b - c) + c^2a(c - a) \geq 0.$$

Determine when equality occurs.

Twenty-third International Olympiad, 1982

1982/1. The function $f(n)$ is defined for all positive integers n and takes on non-negative integer values. Also, for all m, n

$$f(m+n) - f(m) - f(n) = 0 \text{ or } 1$$

$$f(2) = 0, f(3) > 0, \text{ and } f(9999) = 3333.$$

Determine $f(1982)$.

1982/2. A non-isosceles triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3 (a_i is the side opposite A_i). For all $i = 1, 2, 3$, M_i is the midpoint of side a_i, and T_i. is the point where the incircle touches side a_i. Denote by S_i the reflection of T_i in the interior bisector of angle A_i. Prove that the lines M_1, S_1, M_2S_2, and M_3S_3 are concurrent.

1982/3. Consider the infinite sequences $\{x_n\}$ of positive real numbers with the following properties:

$$x_0 = 1, \text{and for all } i \geq 0, x_{i+1} \leq x_i.$$

(a) Prove that for every such sequence, there is an $n \geq 1$ such that

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} \geq 3.999.$$

(b) Find such a sequence for which

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \cdots + \frac{x_{n-1}^2}{x_n} < 4.$$

1982/4. Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers (x, y), then it has at least three such solutions. Show that the equation has no solutions in integers when $n = 2891$.

1982/5. The diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by the inner points M and N, respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

Determine r if B, M, and N are collinear.

1982/6. Let S be a square with sides of length 100, and let L be a path within S which does not meet itself and which is composed of line segments $A_0A_1, A_1A_2, \cdots, A_{n-1}A_n$ with $A_0 \neq A_n$. Suppose that for every point P of the boundary of S there is a point of L at a distance from P not greater than $1/2$. Prove that there are two points X and Y in L such that the distance between X and Y is not greater than 1, and the length of that part of L which lies between X and Y is not smaller than 198.

Twenty-second International Olympiad, 1981

1981/1. P is a point inside a given triangle $ABC.D, E, F$ are the feet of the perpendiculars from P to the lines BC, CA, AB respectively. Find all P for which

$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}$$

is least.

1981/2. Let $1 \leq r \leq n$ and consider all subsets of r elements of the set $\{1, 2, ..., n\}$. Each of these subsets has a smallest member. Let $F(n, r)$ denote the arithmetic mean of these smallest numbers; prove that

$$F(n, r) = \frac{n+1}{r+1}.$$

1981/3. Determine the maximum value of m^3+n^3,where m and n are integers satisfying $m, n \in \{1, 2, ..., 1981\}$ and $(n^2 - mn - m^2)^2 = 1$.

1981/4. (a) For which values of $n > 2$ is there a set of n consecutive positive integers such that the largest number in the set is a divisor of the least common multiple of the remaining $n - 1$ numbers?

(b) For which values of $n > 2$ is there exactly one set having the stated property?

1981/5. Three congruent circles have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle and the point O are collinear.

1981/6. The function $f(x, y)$ satisfies

(1) $f(0, y) = y + 1$,

(2)$f(x + 1, 0) = f(x, 1)$,

(3) $f(x + 1, y + 1) = f(x, f(x + 1, y))$,

for all non-negative integers x, y. Determine $f(4, 1981)$.