Complex Variables

Chapter 17

Functions of a Complex Variable

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Complex Numbers

- Solve the quadratic equation

$$x^2 + x + 1 = 0 \;\Rightarrow x = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\sqrt{-1}}{2}$$

  - Define imaginary unit i by $i^2 = -1$

- Def. A complex number is any number of the form $z = a + ib$ where a & b are real numbers and i is the imaginary unit.
  - Real part → $\mathrm{Re}(z) = a$
  - Imaginary part → $\mathrm{Im}(z) = b$

Complex Numbers

- Def. Complex numbers z_1 & z_2 are equal if $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ & $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.
  - A complex number $z = 0$ if $\mathrm{Re}(z) = 0$ & $\mathrm{Im}(z) = 0$.
- If $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$
  - Addition: $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$
  - Subtraction: $z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$
  - Multiplication:
    $$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(y_1 x_2 + x_1 y_2)$$
  - Division: $$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\,\frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2}$$

Complex Numbers

- Familiar laws hold for complex numbers
  - Commutative laws: $$\begin{cases} z_1 + z_2 = z_2 + z_1 \\ z_1 z_2 = z_2 z_1 \end{cases}$$
  - Associative laws: $$\begin{cases} z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \\ z_1 (z_2 z_3) = (z_1 z_2) z_3 \end{cases}$$
  - Distributive law: $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$

Complex Numbers

- Complex conjugate
  - If $z = x + iy$, then the conjugate of z is $\bar{z} = x - iy$
  - It is very easy to show that

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2 \qquad \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$$

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2 \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

  - The sum & product of a conjugate pair are **real**

$$\begin{cases} z + \bar{z} = (x + iy) + (x - iy) = 2x \cdots (1) \\ z\bar{z} = (x + iy)(x - iy) = x^2 - i^2 y^2 = x^2 + y^2 \cdots (2) \end{cases}$$

Complex Numbers

- Difference between a conjugate pair is **imaginary**

$$z - \bar{z} = (x + iy) - (x - iy) = 2iy \cdots (3)$$

- (1) & (3) yield two useful formulas

$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2} \quad \text{and} \quad \mathrm{Im}(z) = \frac{z - \bar{z}}{2i}$$

- Division of complex numbers using (2)

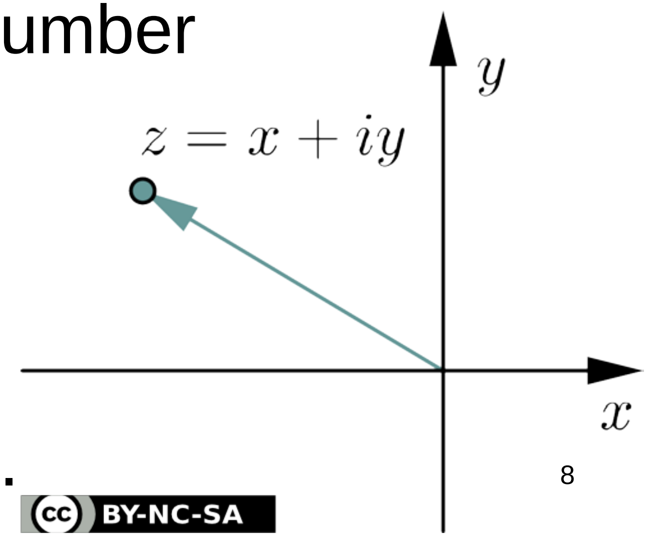
$$\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} = \frac{(x_1 x_2 + y_1 y_2) + i(y_1 x_2 - x_1 y_2)}{x_2^2 + y_2^2}$$

Ex. If $z_1 = 2 - 3i$ & $z_2 = 4 + 6i$, find $\frac{z_1}{z_2}$ & $\frac{1}{z_1}$

Complex Numbers

- **Geometric interpretation**
  - A complex number $z = x + iy$ is uniquely determined by an ordered pair of **real numbers** (x, y).

    Ex. The ordered pair $(2, -3)$ corresponds to the complex number $z = 2 - 3i$.
  - One can associate a complex number $z = x + iy$ with a point (x, y) in a **coordinate plane**.
  - The complex number can also be viewed as a vector from the origin to the terminal point (x, y).



Complex Numbers

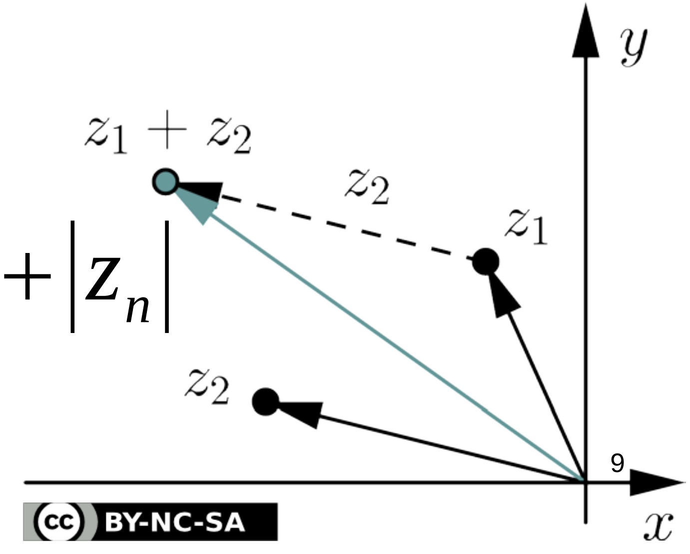
- Def. **Modulus** or **absolute value** of $z = x + iy$, denoted by $|z|$, is the **real number**

$$|z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}} \cdots (4)$$

- Sum of the vectors z_1 & z_2 is the vector $z_1 + z_2$. And we have

$$|z_1 + z_2| \le |z_1| + |z_2| \cdots (5)$$

$$\begin{cases} |z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n| \\ |z_1 + z_2| \ge |z_1| - |z_2| \end{cases}$$



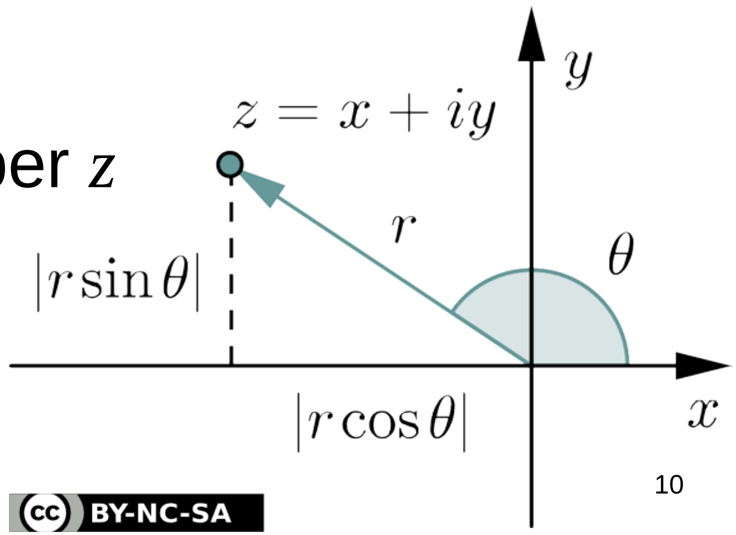
Powers and Roots

- **Polar form**
  - Rectangular coordinate (x, y) and polar coordinate (r, θ) are related by $x = r\cos\theta$ & $y = r\sin\theta$.
  - A nonzero complex number $z = x + iy$ can be written as $z = (r\cos\theta) + i(r\sin\theta)$ or

$$z = r(\cos\theta + i\sin\theta) \cdots (6)$$

  - Polar form of complex number z

$$\begin{cases} r = |z| \\ \theta = \arg(z) \end{cases} \Rightarrow \tan\theta = \frac{y}{x}$$



Powers and Roots

- Argument of a complex number in the interval $-\pi < \theta \le \pi$ is called the **principal argument** of z and is denoted by $\mathrm{Arg}(z)$.

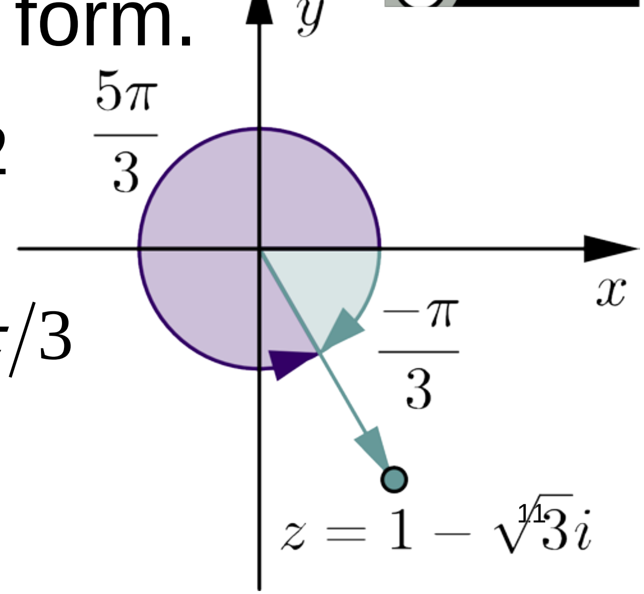
Ex. $\mathrm{Arg}(i) = \pi/2$.

Ex. Express $z = 1 - \sqrt{3}i$ in polar form.

$$\begin{cases} x = 1 \\ y = -\sqrt{3} \end{cases} \Rightarrow r = |z| = \sqrt{(1)^2 + \left(-\sqrt{3}\right)^2} = 2$$

$$\tan\theta = -\sqrt{3}/1 = -\sqrt{3} \;\Rightarrow \theta = \arg(z) = 5\pi/3$$

$$\therefore z = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$



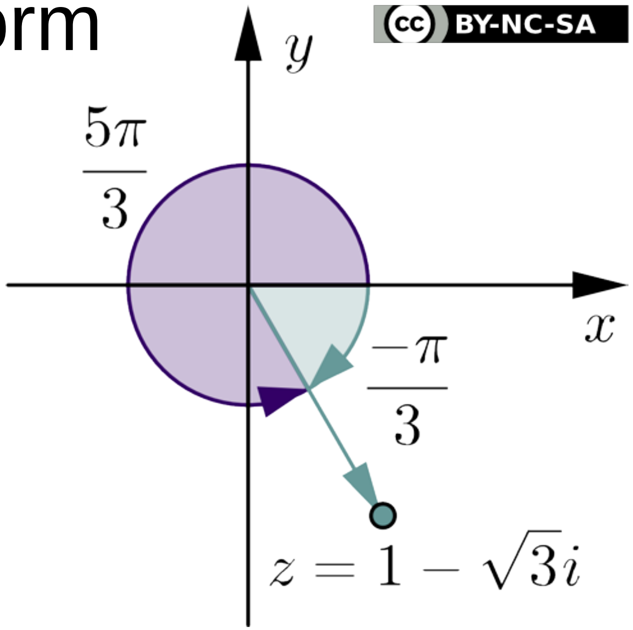
Powers and Roots

- The principal argument of z is

  $\theta = \operatorname{Arg}(z) = -\pi/3$

  Thus, an alternative polar form of the complex number is

  $$z = 2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$$



Powers and Roots

- **Multiplication & division**
  - It is very convenient to use the polar form.
  - Suppose $\begin{cases} z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \\ z_2 = r_2(\cos\theta_2 + i\sin\theta_2) \end{cases}$

$$\Rightarrow z_1 z_2 = r_1 r_2 \left[(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2) \right]$$

$$\therefore z_1 z_2 = r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) \right]$$

$$\text{or } \begin{cases} |z_1 z_2| = |z_1||z_2| \\ \arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \end{cases}$$

Powers and Roots

$$\Rightarrow \frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right)\right.$$

$$\left. + i\left(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2\right)\right]$$

$$\therefore \frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos\left(\theta_1 - \theta_2\right) + i\sin\left(\theta_1 - \theta_2\right)\right]$$

or $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

- It is **NOT** true that $\begin{cases} \text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) \\ \text{Arg}(z_1 / z_2) = \text{Arg}(z_1) - \text{Arg}(z_2) \end{cases}$

Powers and Roots

- **Powers of z**
  - Integer powers of the complex number z

$$z^2 = r^2\left[\cos(\theta+\theta) + i\sin(\theta+\theta)\right] = r^2(\cos 2\theta + i\sin 2\theta)$$

$$z^3 = z^2 z = r^3(\cos 3\theta + i\sin 3\theta)$$

$$\vdots$$

  - Moreover,

$$z^{-2} = \frac{1}{z^2} = r^{-2}\left[\cos(-2\theta) + i\sin(-2\theta)\right]$$

  - For any integer n: $z^n = r^n\left[\cos n\theta + i\sin n\theta\right]$

Powers and Roots

- **Roots**
  - w is said to be an n-th root of z if $w^n = z$
  - Let $w = \rho(\cos\phi + i\sin\phi)$ and $z = r(\cos\theta + i\sin\theta)$

$$\Rightarrow w^n = \rho^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta)$$

$$\Rightarrow \begin{cases} \rho^n = r \Rightarrow \rho = r^{1/n} \\ \cos n\phi = \cos\theta \\ \sin n\phi = \sin\theta \end{cases} \quad n\phi = \theta + 2k\pi \quad \therefore \phi = \frac{\theta + 2k\pi}{n}$$

  - As $k = 0, 1, 2, \ldots, n-1$, we obtain n distinct roots with the same modulus but different arguments.

Powers and Roots

- For $k = n + m$, where $m = 0, 1, 2, \ldots$. Then

$$\phi = \frac{\theta + 2(n+m)\pi}{n} = \frac{\theta + 2m\pi}{n} + 2\pi$$

$$\Rightarrow \sin\phi = \sin\left(\frac{\theta + 2m\pi}{n}\right), \quad \cos\phi = \cos\left(\frac{\theta + 2m\pi}{n}\right)$$

- To summarize, the n-th root of a nonzero complex number $z = r(\cos\theta + i\sin\theta)$ are given by

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

where $k = 0, 1, 2, \cdots, n-1$

Powers and Roots

- Ex. Find the three cube roots of $z = i$.

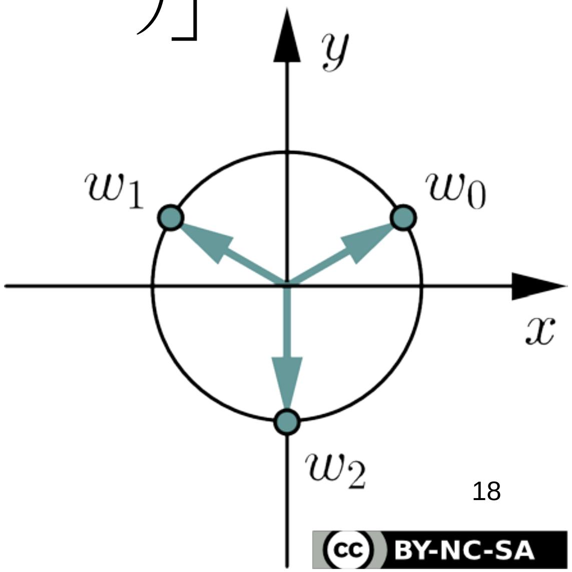
$$\Rightarrow \begin{cases} r = 1 \\ \theta = \arg(i) = \pi/2 \end{cases} \qquad \therefore z = \cos(\pi/2) + i\sin(\pi/2)$$

$$\Rightarrow w_k = (1)^{1/3}\left[\cos\left(\frac{\pi/2 + 2k\pi}{3}\right) + i\sin\left(\frac{\pi/2 + 2k\pi}{3}\right)\right], k = 0, 1, 2.$$

$$k = 0 \Rightarrow w_0 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$k = 1 \Rightarrow w_1 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$k = 2 \Rightarrow w_2 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$$



Powers and Roots

- **Principal *n*-th root of *z***: the root w of z obtained by using the principal argument of z with $k = 0$.
- The n roots of the nonzero z lie on a circle of radius $r^{1/n}$ centered at the origin in the complex plane and are equally spaced on this circle.

Ex. Find the four fourth roots of $z = 1 + i$.

$$\Rightarrow \begin{cases} r = \sqrt{2} \\ \theta = \arg(z) = \pi/4 \end{cases} \qquad \therefore z = \sqrt{2}\left[\cos(\pi/4) + i\sin(\pi/4)\right]$$

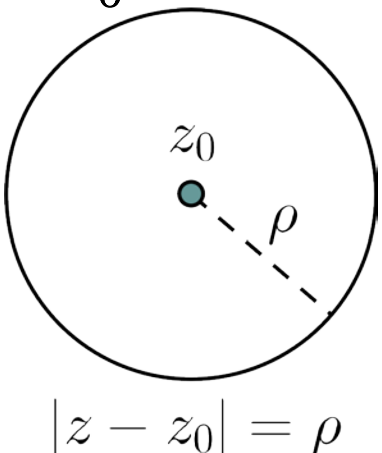
$$\Rightarrow w_k = \left(\sqrt{2}\right)^{1/4}\left[\cos\left(\frac{\pi/4 + 2k\pi}{4}\right) + i\sin\left(\frac{\pi/4 + 2k\pi}{4}\right)\right], k = 0, 1, 2, 3.$$

Sets in Complex Plane

- Since $\left|z-z_0\right|=\sqrt{\left(x-x_0\right)^2+\left(y-y_0\right)^2}$ is the distance between the points $z = x + iy$ and $z_0 = x_0 + iy_0$, the points z satisfying the equation

  $$\left|z-z_0\right|=\rho, \quad (\rho>0)$$

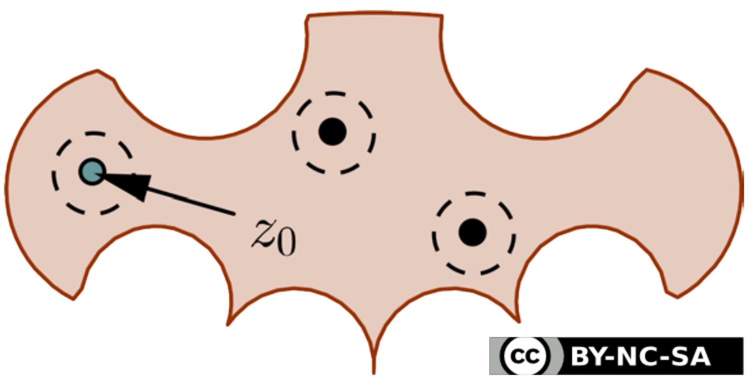
  lie on a **circle** of radius ρ centered at z_0.
  - Ex. $|z| = 1$
  - Ex. $|z - 1 - 2i| = 5$
- The points z satisfying $|z - z_0| < \rho$ lie within, but not on, a circle of radius ρ centered at z_0.

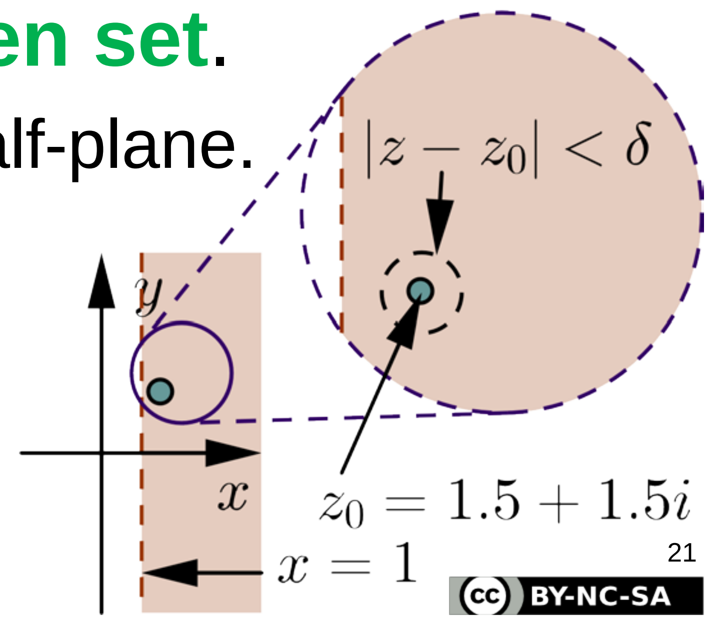


Sets in Complex Plane

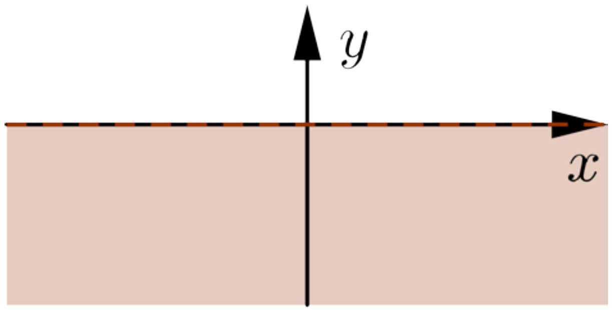
- z_0 is said to be an **interior point** of a set S of the complex plane if there exists some neighborhood of z_0 that lies entirely within S.
- If every point z of a set S is an interior point, then S is said to be an **open set**.

Ex. $\mathrm{Re}(z) > 1$ defines a right half-plane.

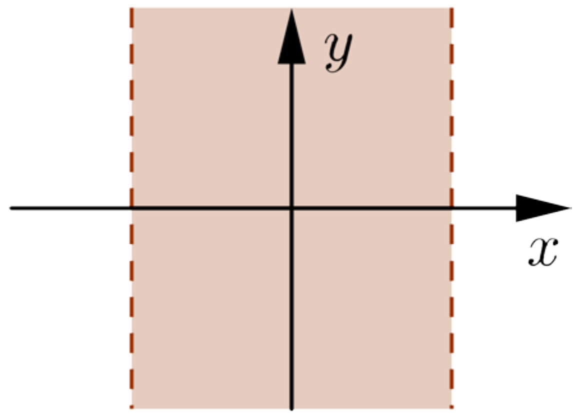




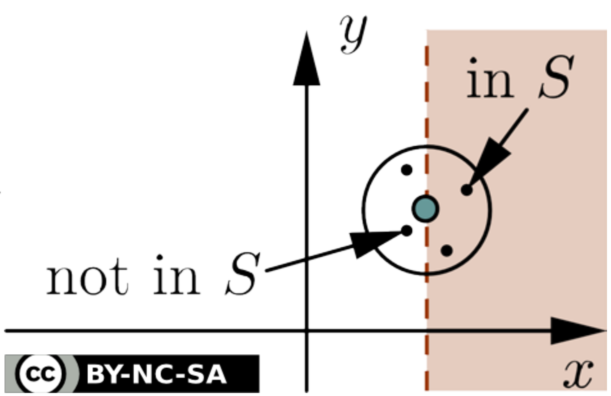
Sets in Complex Plane

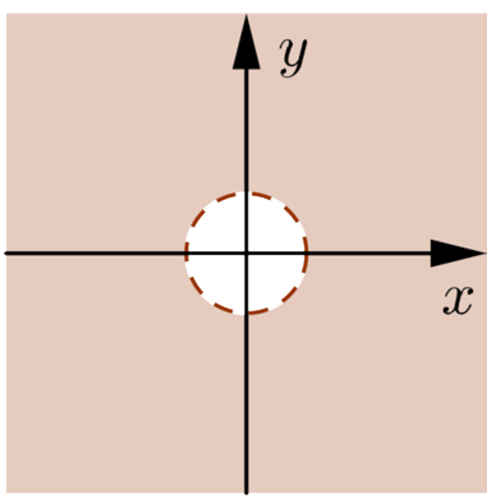


$\mathrm{Im}(z) < 0$

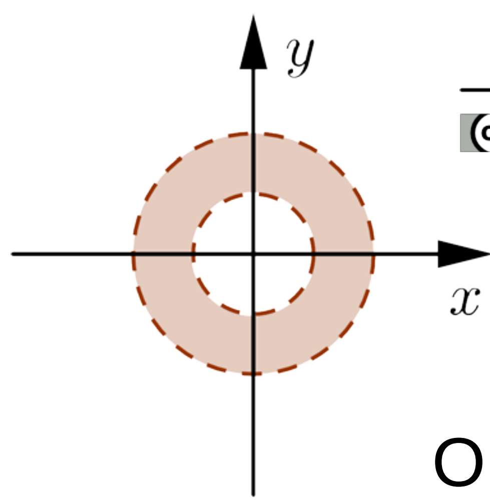


$-1 < \mathrm{Re}(z) < 1$





$|z| > 1$



$1 < |z| < 2$

Open annulus

Sets in Complex Plane

- If every neighborhood of z_0 contains at least one point that is in S & at least one point that is not in S → z_0 is a **boundary point** of S.
- **Boundary** of a set S is the set of all boundary points of S.

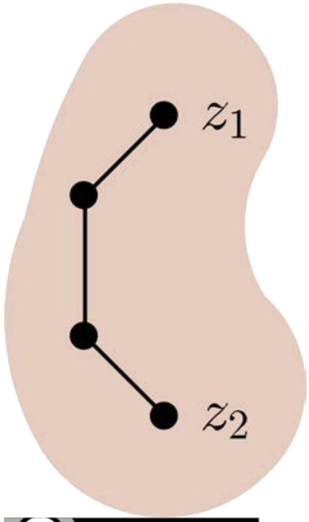
Ex. For $\mathrm{Re}(z) \geq 1$, the points on $\mathrm{Re}(z) = 1$ are boundary points.

Sets in Complex Plane

- If any two points z_1 & z_2 in a set S can be connected by a polygonal line that lies entirely in S → S is a **connected** set.
- An open connected set is called a **domain**.

Ex. The set $\text{Re}(z) \neq 4$ is open but not connected.

- **Region** is a domain in the complex plane with all, some, or none of its boundary points.



Functions of a Complex Variable

- **Function** f from a set A to a set B is a rule of correspondence that assigns to each element in A one and only one element in B.
  - $b = f(a) \Leftrightarrow b$ is the **image** of a
  - **Domain** & **range** of the function f

Ex. A set of real numbers A: $3 \le x < \infty$ & the function given by $f(x) = \sqrt{x-3}$

→ the range of f : $0 \le y < \infty$

→ f is a function of a real variable x.

Functions of a Complex Variable

- When the domain A is a set of complex numbers $z \to f$ is said to be a **function of a complex variable** z.
- The image w of z will be complex, too.

$$w = f(z) = u(x, y) + iv(x, y) \cdots (7)$$

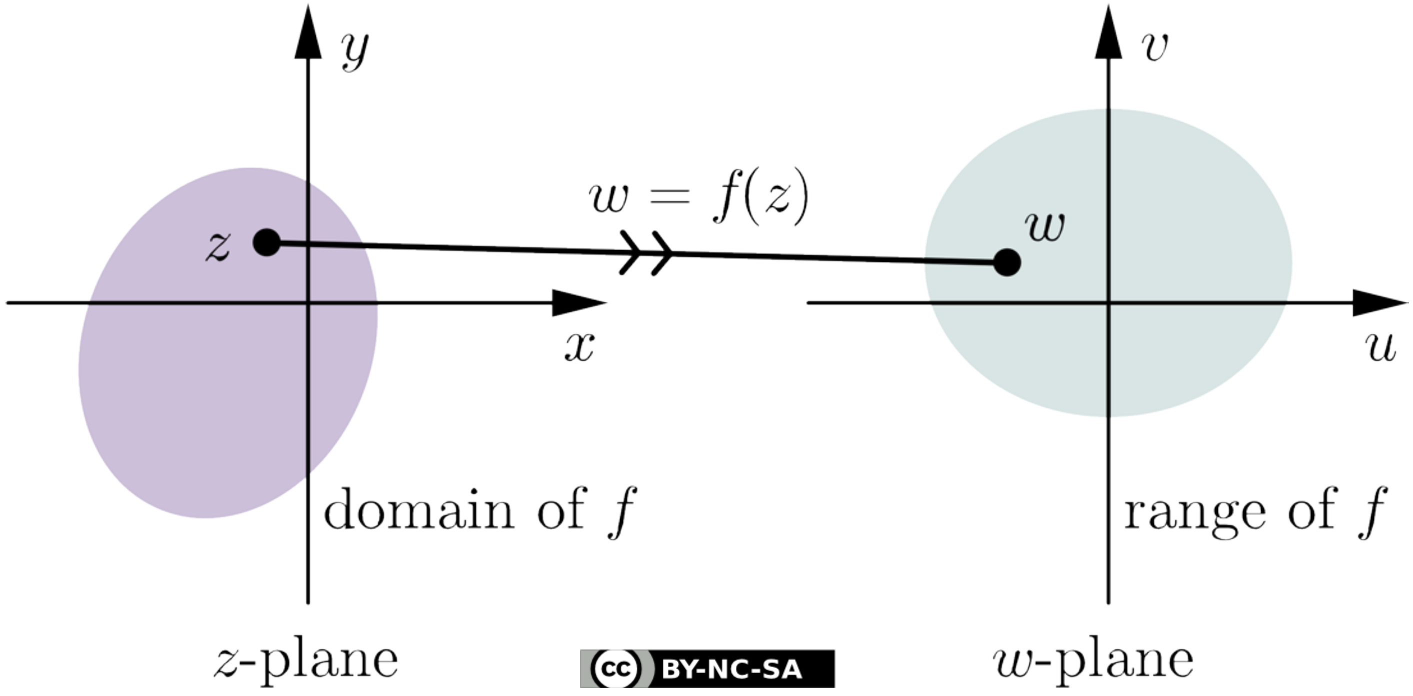
where $u = \mathrm{Re}(w)$ & $v = \mathrm{Im}(w)$ are real-valued functions.

Ex. $f(z) = z^2 - 4z \quad \forall z$

$$\Rightarrow f(z) = (x+iy)^2 - 4(x+iy) = (x^2 - y^2 - 4x) + i(2xy - 4y)$$

Functions of a Complex Variable

- A complex function $w = f(z)$ can be interpreted as a **mapping** or **transformation** from the z-plane to the w-plane.



Functions of a Complex Variable

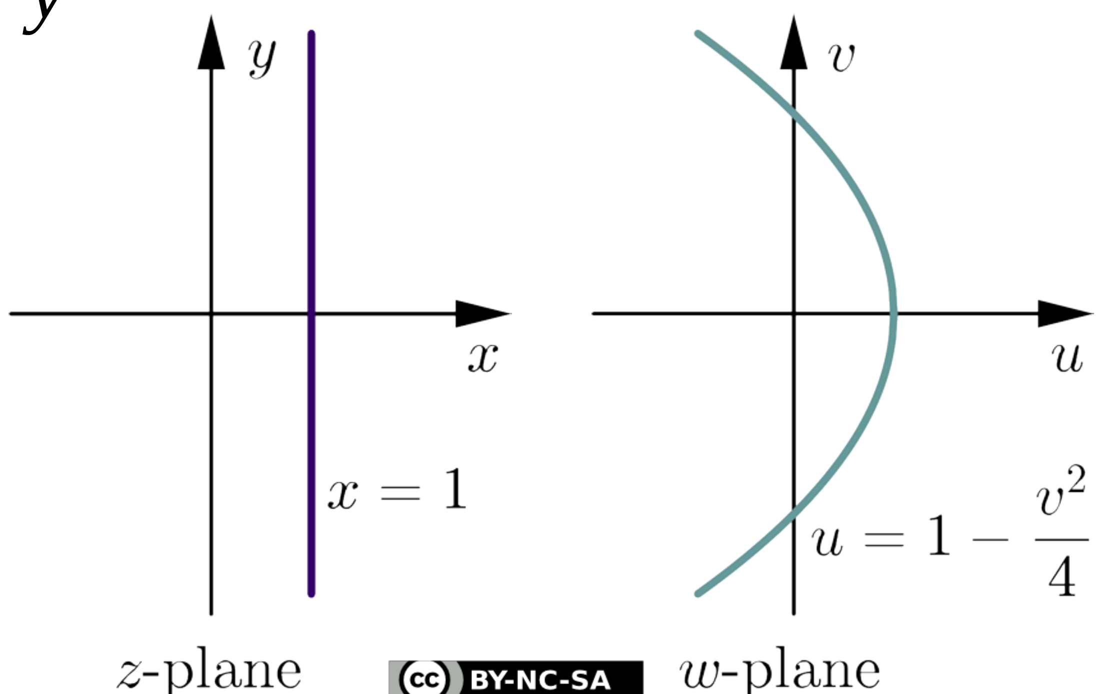
Ex. Find the image of the line $\mathrm{Re}(z) = 1$ under the mapping $f(z) = z^2$.

$$\Rightarrow \begin{cases} u(x,y) = x^2 - y^2 \\ v(x,y) = 2xy \end{cases}$$

$$\because \mathrm{Re}(z) = x = 1$$

$$\Rightarrow \begin{cases} u = 1 - y^2 \\ v = 2y \end{cases}$$

$$\therefore u = 1 - \frac{v^2}{4}$$



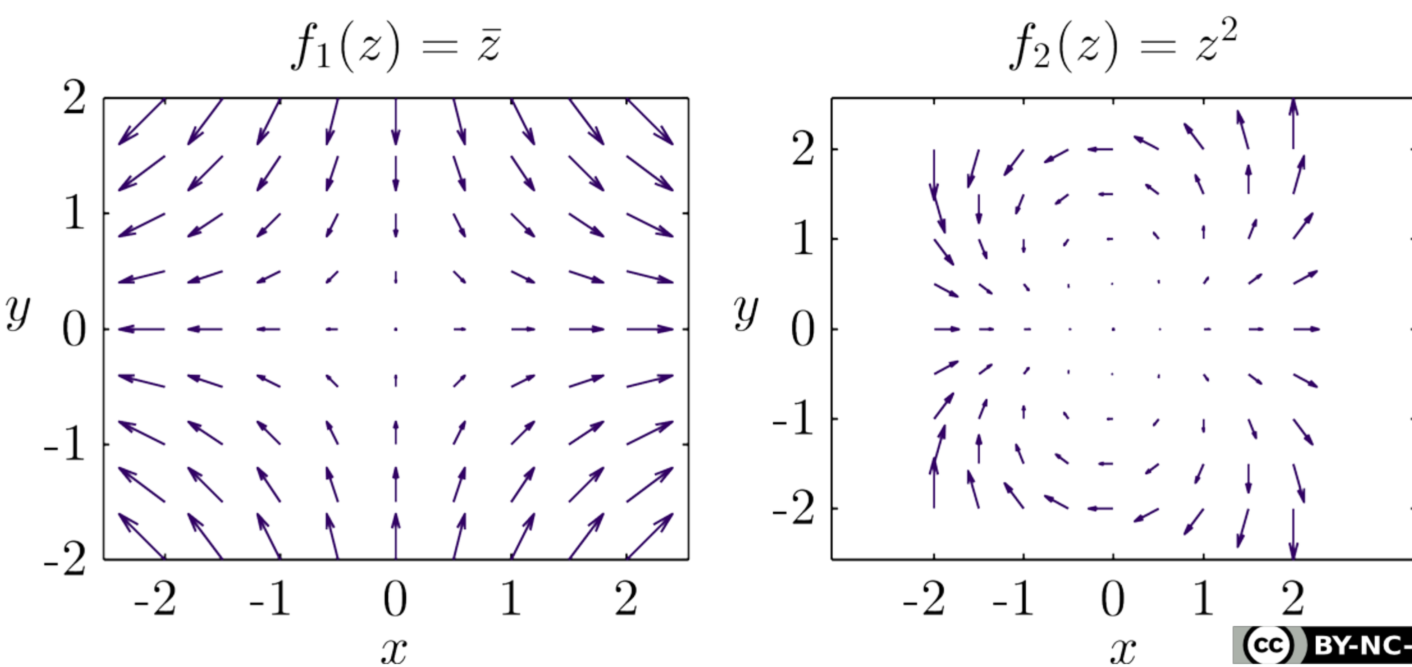
Functions of a Complex Variable

- Complex function $w = f(z)$ is completely determined by real-valued functions $u(x, y)$ & $v(x, y)$, even though $u + iv$ may not be obtainable via familiar operations on z alone.

Ex. $f(z) = xy^2 + i(x^2 - 4y^3)$

Functions of a Complex Variable

- We also may interpret $w = f(z)$ as a 2-D fluid flow by considering $f(z)$ as a vector based at the point z.



Functions of a Complex Variable

- If $x(t) + iy(t)$ is a parametric representation for the path of the flow,

$\vec{T} = \frac{dx(t)}{dt} + i\frac{dy(t)}{dt}$ must coincide with $f\big(x(t)+iy(t)\big)$

- When $f(z) = u(x, y) + iv(x, y)$, the path of the flow satisfies

$$\begin{cases} \frac{dx(t)}{dt} = u(x, y) \\ \frac{dy(t)}{dt} = v(x, y) \end{cases}$$

Functions of a Complex Variable

- Find the streamlines of the flows.

$$f_1(z) = \bar{z} = x - iy \Rightarrow \begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = -y \end{cases} \Rightarrow \begin{cases} x(t) = c_1 e^{t} \\ y(t) = c_2 e^{-t} \end{cases} \Rightarrow xy = c_1 c_2$$

$$f_2(z) = z^2 = \left(x^2 - y^2\right) + i2xy$$

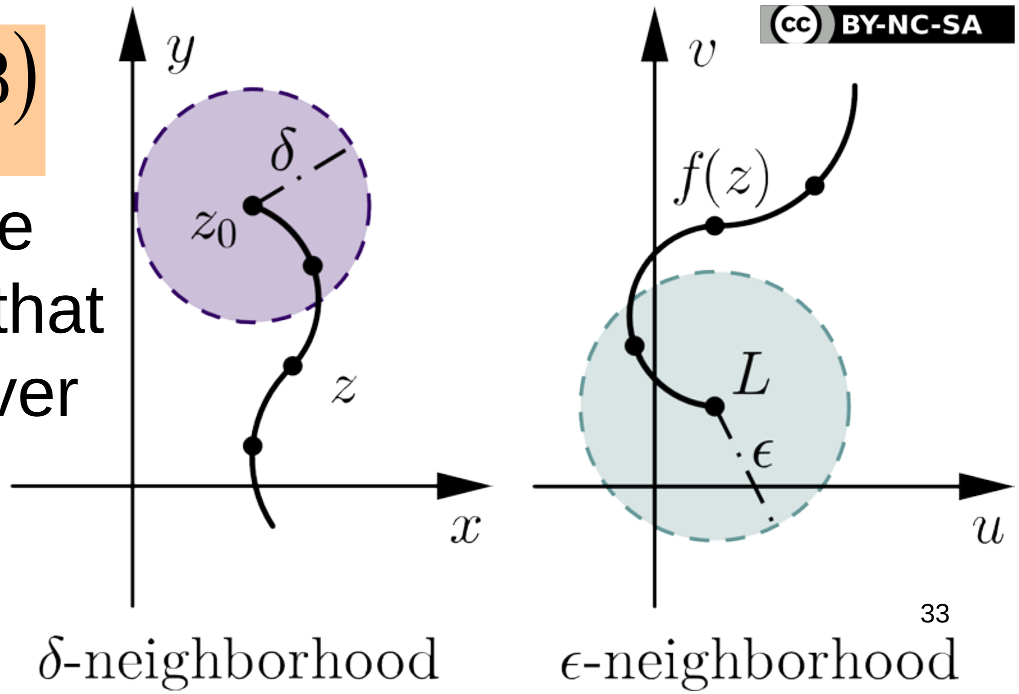
$$\Rightarrow \begin{cases} \dfrac{dx}{dt} = x^2 - y^2 \\ \dfrac{dy}{dt} = 2xy \end{cases} \Rightarrow \frac{dy}{dx} = \frac{2xy}{x^2 - y^2} \Rightarrow x^2 + y^2 = c_2 y$$

Functions of a Complex Variable

- **Limit of a function**
  - Suppose f is defined in some neighborhood of z_0, except possibly at z_0 itself. f is said to possess a limit at z_0,

$$\lim_{z \to z_0} f(z) = L \cdots (8)$$

  - For each $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.



Functions of a Complex Variable

- Limit of sum, product, quotient
  - Suppose $\lim_{z \to z_0} f(z) = L_1$ and $\lim_{z \to z_0} g(z) = L_2$
  - Then

$$(i) \quad \lim_{z \to z_0} \left[f(z) + g(z) \right] = L_1 + L_2$$

$$(ii) \quad \lim_{z \to z_0} f(z) g(z) = L_1 L_2$$

$$(iii) \quad \lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{L_1}{L_2}, \quad L_2 \neq 0$$

Functions of a Complex Variable

- **Continuity at a point**
  - A function f is **continuous** at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$

- A function f defined by

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0, \quad a_n \neq 0$$

where n is a nonnegative integer & a_i ($i = 0$, 1, …, n) are complex constants, is called a **polynomial of degree n**.

  - A polynomial is **continuous everywhere**.

Functions of a Complex Variable

- A **rational function** $f(z) = \dfrac{g(z)}{h(z)}$

  where g & h are polynomial functions, is continuous except at points at which $h(z) = 0$.

- Derivative
  - Suppose f is defined in a neighborhood of z_0.

$$\Rightarrow f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \cdots (9)$$

$$\text{where } \Delta z = \Delta x + i\Delta y$$

Functions of a Complex Variable

- If f is differentiable at z_0, then f is continuous at z_0.
- If f & g are differentiable at a point z, and c is a complex constant, then

$$\frac{d}{dz}c = 0, \quad \frac{d}{dz}cf(z) = cf'(z)$$

$$\frac{d}{dz}[f(z)+g(z)] = f'(z)+g'(z)$$

$$\frac{d}{dz}[f(z)g(z)] = f(z)g'(z)+f'(z)g(z)$$

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)f'(z)-f(z)g'(z)}{[g(z)]^2}$$

$$\frac{d}{dz}f(g(z)) = f'(g(z))g'(z)$$

$$\frac{d}{dz}z^n = nz^{n-1}$$

Functions of a Complex Variable

Ex. Differentiate $f(z) = 3z^4 - 5z^3 + 2z$.

$$\Rightarrow f'(z) = 3 \cdot 4z^3 - 5 \cdot 3z^2 + 2 = 12z^3 - 15z^2 + 2$$

Ex. Differentiate $f(z) = \dfrac{z^2}{4z+1}$

$$\Rightarrow f'(z) = \frac{(4z+1) \cdot 2z - z^2 \cdot 4}{(4z+1)^2} = \frac{4z^2 + 2z}{(4z+1)^2}$$

- In order for f to be differentiable at z_0, (9) must approach the same complex number **from any direction**.

Functions of a Complex Variable

Ex. Show that $f(z) = x + i4y$ is **nowhere differentiable**.

$\Rightarrow$ With $\Delta z = \Delta x + i\Delta y$, we have

$$f(z+\Delta z) - f(z)$$

$$= (x+\Delta x) + 4i(y+\Delta y) - x - 4iy = \Delta x + 4i\Delta y$$

$$\therefore \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta x + 4i\Delta y}{\Delta x + i\Delta y}$$

Let $\Delta z \to 0$ along a line parallel to x-axis

Let $\Delta z \to 0$ along a line parallel to y-axis

Functions of a Complex Variable

- **Analytic functions**
  - A function $f(z)$ is said to be **analytic at a point** z_0 if f is differentiable at z_0 & at every point in some neighborhood of z_0.
  - f is analytic in a domain D if it is analytic at every point in D.

    Ex. $f(z) = |z|^2$ is differentiable only at $z = 0$.

    Ex. $g(z) = z^2$ is differentiable at every point z in the complex plane.
  - A function that is analytic everywhere is said to be an **entire function**.

Functions of a Complex Variable

- A number c is a **zero** of a polynomial function f if and only if $z - c$ is a factor of $f(z)$.

Ex. $f(z) = z^2 - 2z + 2 = (z - 1 - i)(z - 1 + i)$.

Ex. Find the value of the limit.

$$\lim_{z\to 1+i} \frac{z^2 - 2z + 2}{z^2 - 2i} = \lim_{z\to 1+i} \frac{(z-1-i)(z-1+i)}{[z-(1+i)][z-(-1-i)]}$$

$$= \lim_{z\to 1+i} \frac{z-1+i}{z+1+i} = \frac{2i}{2(1+i)} = \frac{1+i}{2}$$

Cauchy-Riemann Equations

- **A necessary condition for analyticity**
  - Suppose $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at z the 1st-order partial derivatives of u & v exist and satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad (10)$$

(Proof)

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x, y) - iv(x, y)}{\Delta x + i\Delta y}$$

Cauchy-Riemann Equations

- Since the limit exists. Δz can approach zero from any direction. In particular, if $\Delta z \to 0$ horizontally,

$$f'(z)=\lim_{\Delta x\to 0}\frac{u(x+\Delta x,y)-u(x,y)}{\Delta x}+i\lim_{\Delta x\to 0}\frac{v(x+\Delta x,y)-v(x,y)}{\Delta x}$$

$$=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\cdots\cdots(11)$$

- If we let $\Delta z \to 0$ vertically,

$$f'(z)=\lim_{\Delta y\to 0}\frac{u(x,y+\Delta y)-u(x,y)}{i\Delta y}+i\lim_{\Delta y\to 0}\frac{v(x,y+\Delta y)-v(x,y)}{i\Delta y}$$

$$=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}\cdots\cdots(12)$$

Cauchy-Riemann Equations

- If f is analytic throughout a domain D, then the real functions u & v must satisfy (10) at every point in D.

Ex. Polynomial $f(z) = z^2 + z$ is analytic for all z.

$$f(z) = (x^2 - y^2 + x) + i(2xy + y) = u(x, y) + iv(x, y)$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x + 1 = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

Ex. Show that $f(z) = (2x^2 + y) + i(y^2 - x)$ is not analytic at any point.

Cauchy-Riemann Equations

$$\Rightarrow \begin{cases} \dfrac{\partial u}{\partial x} = 4x \text{ and } \dfrac{\partial v}{\partial y} = 2y \\ \dfrac{\partial u}{\partial y} = 1 \text{ and } \dfrac{\partial v}{\partial x} = -1 \end{cases}$$

We see that $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ but that $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ is satisfied only on the line $y = 2x$.

→ f is nowhere analytic.

Cauchy-Riemann Equations

- Criterion for analyticity
  - Suppose the real-valued functions u & v are **continuous** and have **continuous 1st-order partial derivatives** in a domain D. If u & v satisfy the **Cauchy-Riemann equations** at all points of D, then $f(z) = u(x, y) + iv(x, y)$ is analytic in D.
  - Ex. For $f(z) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$ we have

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{\left(x^2+y^2\right)^2} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = \frac{-2xy}{\left(x^2+y^2\right)^2} = -\frac{\partial v}{\partial x}$$

Cauchy-Riemann Equations

- (11) & (12) were obtained under the basic assumption that f was differentiable at z.
- Thus, $$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}\cdots(13)$$

Ex. $f(z)=z^2$ is differentiable for all z.

$$\begin{cases} u(x,y)=x^2-y^2 \Rightarrow \dfrac{\partial u}{\partial x}=2x \\ v(x,y)=2xy \Rightarrow \dfrac{\partial v}{\partial x}=2y \end{cases} \qquad \therefore\ f'(z)=2x+i2y=2z$$

Cauchy-Riemann Equations

- If the real-valued functions u & v are continuous & have continuous 1st-order derivatives in a neighborhood of z and if u & v satisfy (10) at z, then $f(z) = u(x, y) + iv(x, y)$ is differentiable at z & $f'(z)$ is given by (13).

Ex. $f(z) = x^2 - iy^2$ is **nowhere analytic**.

$$\Rightarrow \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = -2y, \quad \frac{\partial u}{\partial y} = 0, \quad \frac{\partial v}{\partial x} = 0$$

(10) are satisfied only when $y = -x$. On that line (13) gives $f'(z) = 2x = -2y$.

Cauchy-Riemann Equations

- Harmonic functions
  - A real-valued fx $\phi(x, y)$ that has continuous 2nd-order partial derivatives in a domain D & satisfies Laplace equation is said to be **harmonic** in D.
- If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then u & v are harmonic functions.

(proof)

  - Assume u & v have continuous 2nd-order partial derivatives. Since f is analytic, (10) are satisfied.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

Cauchy-Riemann Equations

- Adding these two equations gives $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
- This shows that $u(x, y)$ is harmonic. Similarly, we can obtain $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

- If $u(x, y)$ is a given fx harmonic in D, it is sometimes possible to find another fx $v(x, y)$ that is harmonic in D so that $u(x, y) + iv(x, y)$ is an analytic fx in D. v is called a **conjugate harmonic function** of u.

Cauchy-Riemann Equations

Ex. (a) Verify that $u(x, y) = x^3 - 3xy^2 - 5y$ is harmonic in the entire complex plane. (b) Find the conjugate harmonic fx of u.

(a) $$\begin{cases} \dfrac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \dfrac{\partial^2 u}{\partial x^2} = 6x \\ \dfrac{\partial u}{\partial y} = -6xy - 5, \quad \dfrac{\partial^2 u}{\partial y^2} = -6x \end{cases}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$$

Cauchy-Riemann Equations

(b) Since v must satisfy (10), we must have

$$\begin{cases} \dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} = 3x^2 - 3y^2 \quad \Rightarrow v(x, y) = 3x^2 y - y^3 + h(x) \\ \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} = -(-6xy - 5) \end{cases}$$

$$\Rightarrow \frac{\partial v}{\partial x} = 6xy + h'(x) = 6xy + 5$$

$$\Rightarrow h'(x) = 5, \quad h(x) = 5x + C$$

$$\therefore v(x, y) = 3x^2 y - y^3 + 5x + C$$

Cauchy-Riemann Equations

- Suppose u & v are the harmonic fxs forming the real & imaginary parts of an analytic fx $f(z)$. The level curves $u(x, y) = c_1$ & $v(x, y) = c_2$ form two orthogonal families of curves.

Ex. $f(z) = z = x + iy \rightarrow x = c_1$ & $y = c_2$.

Exponential & Logarithmic Functions

- Exponential function
  - In real variables, $f(x) = e^x$ has the properties
    $f'(x) = f(x)$ and $f(x_1 + x_2) = f(x_1) f(x_2)$
  - For Euler's formula,
    $e^{iy} = \cos y + i \sin y, \quad y$ a real number
  - For $z = x + iy$, it is natural to expect that
    $e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$
  - The exponential fx of a complex variable z is defined as $e^z = e^{x+iy} = e^x(\cos y + i \sin y) \cdots (14)$

Exponential & Logarithmic Functions

Ex. Evaluate $e^{1.7+4.2i}$.

$$\Rightarrow x = 1.7 \text{ and } y = 4.2$$

$$\Rightarrow e^{1.7+4.2i} = e^{1.7}\left(\cos 4.2 + i\sin 4.2\right) = -2.6837 - 4.7710i$$

- $\mathrm{Re}(e^z) = u(x,y) = e^x\cos y$ & $\mathrm{Im}(e^z) = v(x,y) = e^x\sin y$ are continuous & have continuous 1st partial derivatives at every point z of the complex plane. Moreover, (10) are satisfied at all points.

$$\Rightarrow \frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$$

Exponential & Logarithmic Functions

- Thus, $f(z) = e^z$ is analytic for all z; in other words, f is an **entire function**.
- The derivative of f can be obtained via (11).

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = e^x \cos y + i\left(e^x \sin y\right) = f(z)$$

$$\therefore \frac{d}{dz}e^z = e^z$$

- If $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$, we can have

$$f(z_1)f(z_2) = e^{x_1}(\cos y_1 + i\sin y_1)e^{x_2}(\cos y_2 + i\sin y_2)$$

Exponential & Logarithmic Functions

$$f(z_1)f(z_2) = e^{x_1+x_2}\left[(\cos y_1 \cos y_2 - \sin y_1 \sin y_2) + i(\sin y_1 \cos y_2 + \cos y_1 \sin y_2)\right]$$

$$= e^{x_1+x_2}\left[\cos(y_1+y_2) + i\sin(y_1+y_2)\right]$$

$$= f(z_1+z_2)$$

$$\therefore e^{z_1}e^{z_2} = e^{z_1+z_2}$$

- Similarly, one can prove that $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$

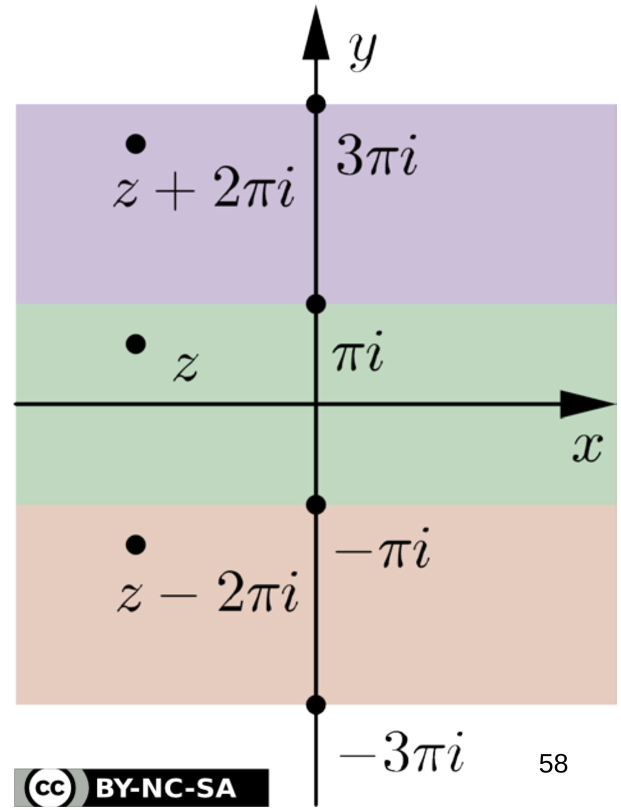
Exponential & Logarithmic Functions

- **Periodicity**
  - $f(z) = e^z$ is **periodic** with the complex period $2\pi i$.

$$\because e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$$

$$\Rightarrow e^{z+2\pi i} = e^z e^{2\pi i} = e^z \quad \text{for all } z$$

$$\therefore f(z+2\pi i) = f(z)$$



  - Divide the complex plane into

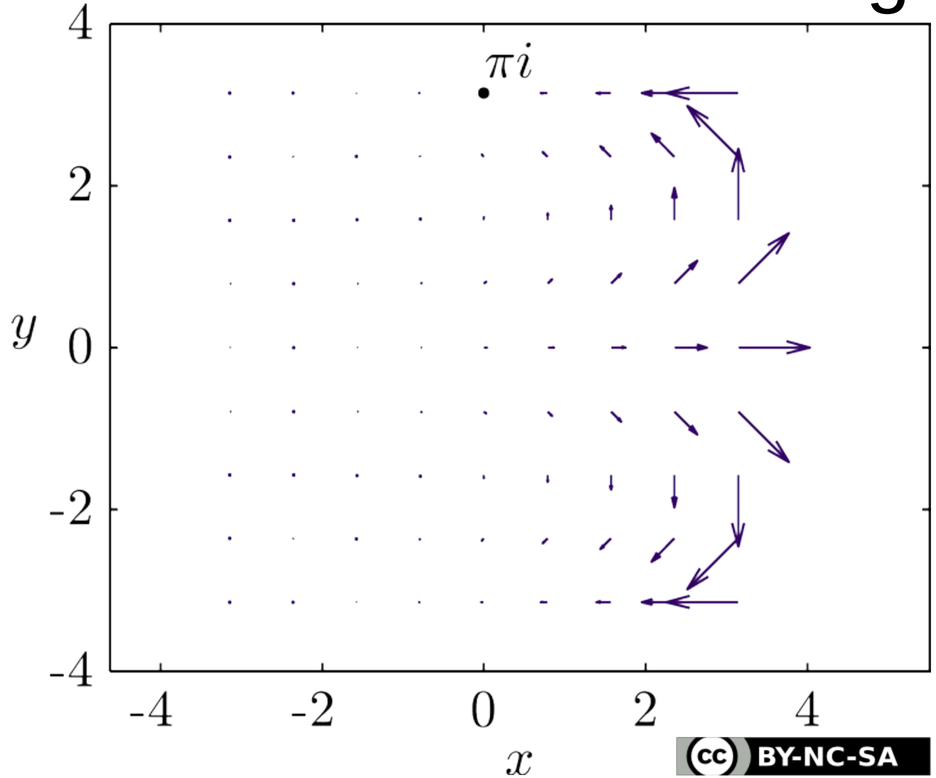
$$(2n-1)\pi < y \le (2n+1)\pi$$

$$\text{where } n = 0, \pm 1, \pm 2, \cdots$$

$$f(z) = f(z \pm 2\pi i) = f(z \pm 4\pi i) = \cdots$$

Exponential & Logarithmic Functions

- The strip $-\pi < y \le \pi$ is called the **fundamental region** for $f(z) = e^z$.
- The flow over the fundamental region.



Exponential & Logarithmic Functions

- **Polar form of a complex number**
  - Using (6), $z = r(\cos\theta + i\sin\theta)$. $\because e^{i\theta} = \cos\theta + i\sin\theta$

$$\therefore z = re^{i\theta}$$

Ex. Find the steady-state current $I(t)$ in an RLC series circuit.

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E_0\sin\omega t \text{ and } I = \frac{dq}{dt}$$

$$\Rightarrow L\frac{dI}{dt} + RI + \frac{1}{C}q = \mathrm{Im}\left(E_0 e^{j\omega t}\right)$$

Exponential & Logarithmic Functions

Assume $I(t) = \mathrm{Im}(I_0 e^{j\omega t})$.

$$\Rightarrow \left(j\omega L + R + \frac{1}{j\omega C} \right) I_0 = E_0$$

$$\therefore I_0 = \frac{E_0}{j\omega L + R + \frac{1}{j\omega C}} = \frac{E_0}{R + j\left(\omega L - \frac{1}{\omega C} \right)} = \frac{E_0}{Z}$$

$$\text{where } Z = |Z| e^{j\theta} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C} \right)^2} e^{j \tan^{-1}\left[\left(\omega L - \frac{1}{\omega C} \right) / R \right]}$$

$$\therefore I(t) = \mathrm{Im}\left(\frac{E_0}{|Z|} e^{-j\theta} e^{j\omega t} \right)$$

Exponential & Logarithmic Functions

- **Logarithmic function**
  - Logarithm of a complex number z $(z \neq 0)$ is defined as the inverse of the exponential function.

$$w = \ln z \quad \text{if } z = e^{w} \quad \cdots (15)$$

  - To find the real & imaginary parts of $\ln z$

$$z = x + iy = e^{w} = e^{u+iv} = e^{u}\left(\cos v + i \sin v\right)$$

$$\Rightarrow x = e^{u} \cos v \;\; \text{and} \;\; y = e^{u} \sin v$$

$$\Rightarrow \begin{cases} x^2 + y^2 = e^{2u} \Rightarrow |z|^2 = e^{2u} \quad \therefore u = \log_e |z| \\ \dfrac{y}{x} = \tan v \Leftrightarrow v = \theta = \arg z \end{cases}$$

Exponential & Logarithmic Functions

- For $z \neq 0$ and $\theta = \arg(z)$

$$\ln z = \log_e |z| + i(\theta + 2n\pi) \text{ for } n = 0, \pm 1, \pm 2, \ldots \quad \cdots (16)$$

  - Note that there are **infinitely many values of the logarithm of a complex number** z.

Ex. Evaluate (a) $\ln(-2)$ and (b) $\ln(-1 - i)$.

$$(a)\ \theta = \arg(-2) = \pi \text{ and } \log_e |-2| = 0.6932$$

$$\therefore \ln(-2) = 0.6932 + i(\pi + 2n\pi)$$

$$(b)\ \theta = \arg(-1-i) = 5\pi/4 \text{ and } \log_e |-1-i| = \log_e \sqrt{2} = 0.3466$$

$$\therefore \ln(-1-i) = 0.3466 + i(5\pi/4 + 2n\pi)$$

Exponential & Logarithmic Functions

- Principal value
  - As a consequence of (16), the logarithm of a positive real number has many values.
  - With the principal argument of a complex number, Arg(z), in the interval (−π, π], we can define the **principal value** of $\ln z$ as

$$\text{Ln}\, z = \log_e |z| + i\,\text{Arg}\, z \cdots (17)$$

Ex. Evaluate (a) Ln(−2) & (b) Ln(−1 − *i*).

$$(a)\ \theta = \text{Arg}(-2) = \pi \quad \therefore \text{Ln}(-2) = 0.6932 + \pi i$$

$$(b)\ \theta = \text{Arg}(-1-i) = -3\pi/4 \quad \therefore \text{Ln}(-1-i) = 0.3466 - i(3\pi/4)$$

Exponential & Logarithmic Functions

- (16) can be interpreted as an infinite collection of logarithmic functions. Each fx in the collection is called a **branch** of $\ln z$.
- $f(z) = \text{Ln}\, z$ is called the **principal branch** of $\ln z$ or the **principal logarithmic function**.
- Some familiar properties hold in the complex case:

$$\begin{cases} \ln(z_1 z_2) = \ln z_1 + \ln z_2 \\ \ln\left(\dfrac{z_1}{z_2}\right) = \ln z_1 - \ln z_2 \end{cases}$$

Exponential & Logarithmic Functions

Ex. For $z_1 = i$ & $z_2 = -1 + i$,

$$\mathbf{Ln}(z_1 z_2) = \mathbf{Ln}(-1-i) = 0.3466 - i\frac{3\pi}{4}$$

$$\mathbf{Ln}z_1 + \mathbf{Ln}z_2 = \left(0 + i\frac{\pi}{2}\right) + \left(0.3466 + i\frac{3\pi}{4}\right)$$

$$= 0.3466 + i\frac{5\pi}{4} \neq \mathbf{Ln}(z_1 z_2)$$

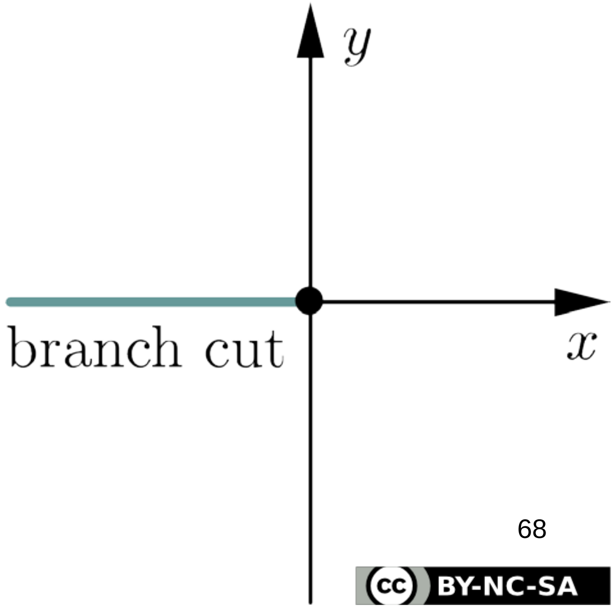
Exponential & Logarithmic Functions

- Analyticity
  - $f(z) = \mathrm{Ln}\, z$ is not continuous at $z = 0$ since $f(0)$ is not defined.
  - $f(z) = \mathrm{Ln}\, z$ is discontinuous at all points of the negative real axis because $\mathrm{Im}[f(z)] = v = \mathrm{Arg}(z)$ is discontinuous at these points.
    - For x_0 on the negative real axis, as $z \to x_0$ from the upper half-plane, $\mathrm{Arg}(z) \to \pi$, whereas as $z \to x_0$ from the lower half-plane, $\mathrm{Arg}(z) \to -\pi$.
  - Thus, $f(z) = \mathrm{Ln}\, z$ is **NOT analytic on the nonpositive real axis**.

Exponential & Logarithmic Functions

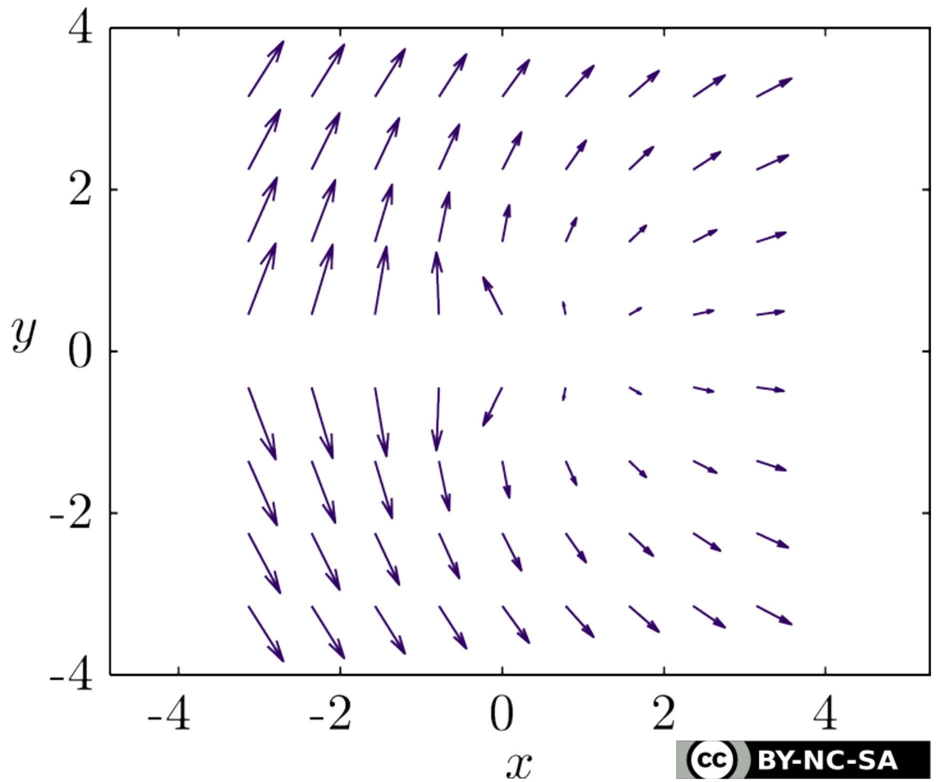
- However, $f(z) = \text{Ln}\, z$ is analytic throughout the domain D consisting of all the points in the complex plane except the nonpositive real axis.
- Since $f(z) = \text{Ln}\, z$ is the principal branch of $\ln z$, the nonpositive real axis is referred to as a **branch cut** for the function.
- (10) is satisfied throughout D.
- Also,

$$\frac{d}{dz}\text{Ln} z = \frac{1}{z} \quad \text{for all } z \text{ in } D$$



Exponential & Logarithmic Functions

- The figure shows $w = \text{Ln}\, z$ as a flow.



Exponential & Logarithmic Functions

- Complex powers
  - Define complex powers of a complex number.
  - If α is a complex number & $z = x + iy$,

$$z^{\alpha} = e^{\alpha \ln z} \quad \text{for } z \neq 0 \qquad \cdots(18)$$

  - Since $\ln z$ is multiple-valued, z^{α} is **multiple-valued**.
  - However, when $\alpha = n$ (integer), (18) is **single-valued** since there is only one value for z^2, z^3, z^{-1}…

Ex. Suppose $\alpha = 2$ & $z = re^{i\theta}$

$$e^{2\ln z} = e^{2(\log_e r + i(\theta + 2k\pi))} = e^{2\log_e r} e^{2i\theta} e^{i4k\pi} = r^2 e^{i\theta} e^{i\theta} \cdot 1$$

$$= re^{i\theta} \cdot re^{i\theta} = z^2$$

Exponential & Logarithmic Functions

- If we use $\mathrm{Ln}\, z$ in place of $\ln z$, (18) gives the **principal values** of z^{α}.

Ex. Evaluate i^{2i}.

$$z = i,\ \arg z = \pi/2,\ \alpha = 2i$$

$$\Rightarrow i^{2i} = e^{2i[\log_e 1 + i(\pi/2 + 2n\pi)]} = e^{-(1+4n)\pi},\quad n = 0, \pm 1, \pm 2, \cdots$$

i^{2i} is real for every value of n.

Since $\mathrm{Arg}(z) = \pi/2$, we obtain the principal value of i^{2i} for $n = 0$.

$$\Rightarrow i^{2i} = e^{-\pi} \cong 0.043$$

Trigonometric & Hyperbolic Functions

- Trigonometric functions
  - For a real variable x,
    $$e^{ix} = \cos x + i \sin x \text{ and } e^{-ix} = \cos x - i \sin x$$
    $$\Rightarrow \sin x = \frac{e^{ix} - e^{-ix}}{2i} \text{ and } \cos x = \frac{e^{ix} + e^{-ix}}{2}$$
  - Similarly, for a complex number $z = x + iy$,
    $$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \text{ and } \cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \cdots (19)$$

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{1}{\tan z}, \quad \sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}$$

Trigonometric & Hyperbolic Functions

- **Analyticity**
  - Since e^{iz} & e^{-iz} are entire functions, it follows that $\sin z$ & $\cos z$ are **entire functions**.
  - Note that $\sin z = 0$ only for $z = n\pi$ & $\cos z = 0$ only for $z = (2n + 1)\pi/2$. Thus, $\tan z$ & $\sec z$ are analytic except at $z = (2n + 1)\pi/2$, and $\cot z$ & $\csc z$ are analytic except at $z = n\pi$.

Trigonometric & Hyperbolic Functions

- Derivatives
  - Since $(d/dz)e^z = e^z$, we have $(d/dz)e^{iz} = ie^{iz}$ and $(d/dz)e^{-iz} = -ie^{-iz}$.

$$\Rightarrow \frac{d}{dz}\sin z = \frac{d}{dz}\frac{e^{iz}-e^{-iz}}{2i} = \frac{e^{iz}+e^{-iz}}{2} = \cos z$$

$$\frac{d}{dz}\sin z = \cos z \qquad \frac{d}{dz}\cos z = -\sin z$$

$$\frac{d}{dz}\tan z = \sec^2 z \qquad \frac{d}{dz}\cot z = -\csc^2 z \qquad \cdots(20)$$

$$\frac{d}{dz}\sec z = \sec z \tan z \qquad \frac{d}{dz}\csc z = -\csc z \cot z$$

Trigonometric & Hyperbolic Functions

- Identities
  - Same in the complex case.

$$\sin(-z) = -\sin z$$

$$\cos(-z) = \cos z$$

$$\cos^2 z + \sin^2 z = 1$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\sin 2z = 2 \sin z \cos z$$

$$\cos 2z = \cos^2 z - \sin^2 z$$

Trigonometric & Hyperbolic Functions

- If y is real, the hyperbolic sine & cosine are

$$\sinh y = \frac{e^{y} - e^{-y}}{2} \text{ and } \cosh y = \frac{e^{y} + e^{-y}}{2}$$

- From (19) & Euler's formula

$$\Rightarrow \sin z = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$

$$= \sin x\left(\frac{e^{y} + e^{-y}}{2}\right) + i\cos x\left(\frac{e^{y} - e^{-y}}{2}\right)$$

$$\therefore \begin{cases} \sin z = \sin x \cosh y + i \cos x \sinh y \\ \cos z = \cos x \cosh y - i \sin x \sinh y \end{cases} \cdots (21)$$

Trigonometric & Hyperbolic Functions

- From (21),

$$1 = \cosh^2 y - \sinh^2 y \cdots (22)$$

$$\begin{aligned} |\sin z|^2 &= \sin^2 x \cdot \cosh^2 y + \cos^2 x \cdot \sinh^2 y \\ &= \sin^2 x(1 + \sinh^2 y) + \cos^2 x \cdot \sinh^2 y \\ &= \sin^2 x + \sinh^2 y \cdots (23) \end{aligned}$$

$$|\cos z|^2 = \cos^2 x + \sinh^2 y \cdots (24)$$

Trigonometric & Hyperbolic Functions

- **Zeros**
  - A complex number z is zero iff $|z|^2 = 0$.
  - To have $\sin z = 0$, we must have $\sin^2 x + \sinh^2 y = 0$. from (23). This implies that $\sin x = 0$ & $\sinh y = 0$, and so $x = n\pi$ & $y = 0$.

  → Zeros of $\sin z$ are $z = x + iy = n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$

  - Similarly, zeros of $\cos z$ are $z = (2n + 1)\pi/2$, where $n = 0, \pm 1, \pm 2, \ldots$

Trigonometric & Hyperbolic Functions

Ex. Evaluate $\sin(2 + i)$.

$$\Rightarrow \sin(2+i) = \sin 2 \cosh 1 + i \cos 2 \sinh 1 = 1.4031 - 0.4891i$$

Ex. Solve the equation $\cos z = 10$.

$$\Rightarrow \cos z = \frac{e^{iz} + e^{-iz}}{2} = 10$$

$$\Rightarrow e^{2iz} - 20e^{iz} + 1 = 0$$

$$\Rightarrow e^{iz} = 10 \pm 3\sqrt{11}$$

$$\Rightarrow iz = \log_e\left(10 \pm 3\sqrt{11}\right) + 2n\pi i \quad \text{for } n = 0, \pm 1, \pm 2, \cdots$$

$$\therefore z = 2n\pi \mp i \log_e\left(10 + 3\sqrt{11}\right) \quad \text{for } n = 0, \pm 1, \pm 2, \cdots$$

Trigonometric & Hyperbolic Functions

- Hyperbolic sine & cosine
  - For any complex number $z = x + iy$,

$$\sinh z = \frac{e^z - e^{-z}}{2} \text{ and } \cosh z = \frac{e^z + e^{-z}}{2} \qquad \cdots(25)$$

  - Also,

$$\tanh z = \frac{\sinh z}{\cosh z} \qquad \coth z = \frac{1}{\tanh z}$$
$$\operatorname{sech} z = \frac{1}{\cosh z} \qquad \operatorname{csch} z = \frac{1}{\sinh z} \qquad \cdots(26)$$

Trigonometric & Hyperbolic Functions

- Hyperbolic sine & cosine are **entire functions**.
- Functions of (26) are analytic except at points where the denominators are zero.
- From (25), it is easy to see that

$$\frac{d}{dz}\sinh z = \cosh z \quad \text{and} \quad \frac{d}{dz}\cosh z = \sinh z \quad \cdots(27)$$

- Trigonometric & hyperbolic functions are related in complex calculus.

$$\sin z = -i\sinh(iz), \quad \cos z = \cosh(iz) \quad \cdots(28)$$

$$\sinh z = -i\sin(iz), \quad \cosh z = \cos(iz) \quad \cdots(29)$$

Trigonometric & Hyperbolic Functions

- **Zeros**
  - Zeros of $\sinh z$ & $\cosh z$ are **pure imaginary** and are respectively,
    $z = n\pi i$ and $z = (2n+1)\frac{\pi i}{2}$ for $n = 0, \pm 1, \pm 2, \cdots$
  - Also, note that

$$\sinh z = -i\sin(iz) = -i\sin(-y+ix)$$

$$= -i[\sin(-y)\cosh x + i\cos(-y)\sinh x]$$

$$= -i[-\sin y\cosh x + i\cos y\sinh x]$$

$$\therefore \sinh z = \sinh x\cos y + i\cosh x\sin y \cdots (30)$$

Similarly, $\cosh z = \cosh x\cos y + i\sinh x\sin y \cdots (31)$

Trigonometric & Hyperbolic Functions

- **Periodicity**
  - From (21),

$$\sin(z+2\pi) = \sin(x+2\pi+iy)$$
$$= \sin(x+2\pi)\cosh y + i\cos(x+2\pi)\sinh y$$
$$= \sin x\cosh y + i\cos x\sinh y = \sin z$$

$$\cos(z+2\pi) = \cos z$$

  - From (30) & (31),

$$\sinh(z+2\pi i) = \sinh(x+iy+2\pi i)$$
$$= \sinh x\cos(y+2\pi) + i\cosh x\sin(y+2\pi) = \sinh z$$

$$\cosh(z+2\pi i) = \cosh z$$

Inverse Trigonometric & Hyperbolic Functions

- Since the inverse of these analytic functions are **multiple-valued functions**, they do **NOT** possess inverse functions in its strictest interpretation.
- **Inverse sine**
  - Def. $w = \sin^{-1} z \quad \text{if } z = \sin w$

$$\Rightarrow \frac{e^{iw} - e^{-iw}}{2i} = z \quad \Rightarrow e^{2iw} - 2ize^{iw} - 1 = 0$$

$$\Rightarrow e^{iw} = iz + \left(1 - z^2\right)^{1/2} \quad \therefore \sin^{-1} z = -i \ln\left[iz + \left(1 - z^2\right)^{1/2}\right]$$

Inverse Trigonometric & Hyperbolic Functions

- **Inverse cosine**

$$\frac{e^{iw}+e^{-iw}}{2}=z \quad \Rightarrow e^{2iw}-2ze^{iw}+1=0$$

$$\Rightarrow e^{iw}=z+\left(z^2-1\right)^{1/2} \quad \therefore \cos^{-1} z=-i\ln\left[z+i\left(1-z^2\right)^{1/2}\right]$$

- **Inverse tangent**

$$\Rightarrow \frac{e^{iw}-e^{-iw}}{i\left(e^{iw}+e^{-iw}\right)}=z \quad \Rightarrow e^{2iw}-1=iz\left(e^{2iw}+1\right)$$

$$\Rightarrow e^{2iw}=\frac{1+iz}{1-iz} \quad \therefore \tan^{-1} z=\frac{-i}{2}\ln\frac{i-z}{i+z}=\frac{i}{2}\ln\frac{i+z}{i-z}$$

Inverse Trigonometric & Hyperbolic Functions

Ex. Find all values of $\sin^{-1}\sqrt{5}$

$$\sin^{-1}\sqrt{5} = -i\ln\left[\sqrt{5}i + \left(1-\left(\sqrt{5}\right)^2\right)^{1/2}\right]$$

$$= -i\ln\left[\sqrt{5}i \pm 2i\right] = -i\ln\left[\left(\sqrt{5}\pm 2\right)i\right]$$

$$= -i\left[\log_e\left(\sqrt{5}\pm 2\right) + \left(\frac{\pi}{2}+2n\pi\right)i\right], \quad n = 0, \pm 1, \pm 2, \ldots$$

$$= \frac{\pi}{2} + 2n\pi \mp i\log_e\left(\sqrt{5}+2\right)$$

Inverse Trigonometric & Hyperbolic Functions

- **Derivatives**
  - To find the derivative of $w = \sin^{-1} z$, we begin by differentiating $z = \sin w$:

$$\frac{d}{dz} z = \frac{d}{dz} \sin w \quad \Rightarrow \frac{dw}{dz} = \frac{1}{\cos w} = \frac{1}{\left(1 - \sin^2 w\right)^{1/2}} = \frac{1}{\left(1 - z^2\right)^{1/2}}$$

$$\therefore \frac{d}{dz} \sin^{-1} z = \frac{1}{\left(1 - z^2\right)^{1/2}}$$

$$\frac{d}{dz} \cos^{-1} z = \frac{-1}{\left(1 - z^2\right)^{1/2}} \quad \text{and} \quad \frac{d}{dz} \tan^{-1} z = \frac{1}{1 + z^2}$$

Inverse Trigonometric & Hyperbolic Functions

Ex. Find the derivative of $w = \sin^{-1} z$ at $z = \sqrt{5}$

$$\Rightarrow \left.\frac{dw}{dz}\right|_{z=\sqrt{5}} = \frac{1}{\left(1-\left(\sqrt{5}\right)^2\right)^{1/2}} = \frac{1}{(-4)^{1/2}} = \frac{1}{\pm 2i} = \frac{\mp i}{2}$$

Inverse Trigonometric & Hyperbolic Functions

- Inverse hyperbolic functions & derivatives

$$\sinh^{-1} z = \ln\left[z + \left(z^2+1\right)^{1/2}\right] \qquad \frac{d}{dz}\sinh^{-1} z = \frac{1}{\left(z^2+1\right)^{1/2}}$$

$$\cosh^{-1} z = \ln\left[z + \left(z^2-1\right)^{1/2}\right] \qquad \frac{d}{dz}\cosh^{-1} z = \frac{1}{\left(z^2-1\right)^{1/2}}$$

$$\tanh^{-1} z = \frac{1}{2}\ln\frac{1+z}{1-z} \qquad \frac{d}{dz}\tanh^{-1} z = \frac{1}{1-z^2}$$

Ex. Find all values of $\cosh^{-1}(-1)$

$$\cosh^{-1}(-1) = \ln(-1) = \log_e 1 + (\pi + 2n\pi)i$$

$$= (2n+1)\pi i, \quad n = 0, \pm 1, \pm 2, \ldots$$

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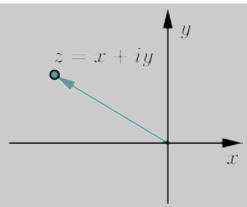
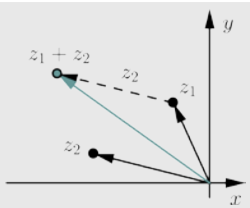
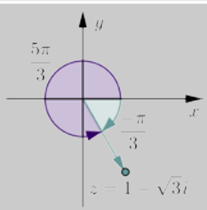
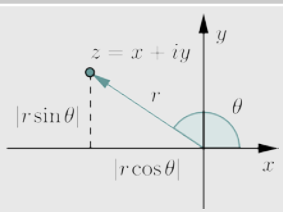
8		CC BY NC SA	Shih-Yuan Chen
9		CC BY NC SA	Shih-Yuan Chen
10		CC BY NC SA	Shih-Yuan Chen
11 12		CC BY NC SA	Shih-Yuan Chen

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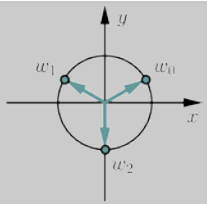
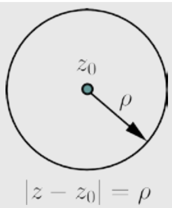
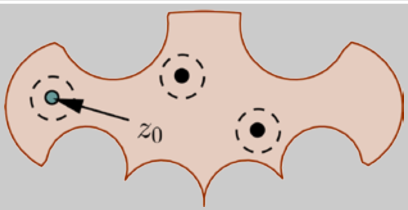
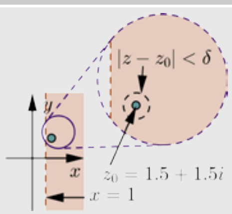
18		CC BY NC SA	Shih-Yuan Chen
20		CC BY NC SA	Shih-Yuan Chen
21		CC BY NC SA	Shih-Yuan Chen
21		CC BY NC SA	Shih-Yuan Chen

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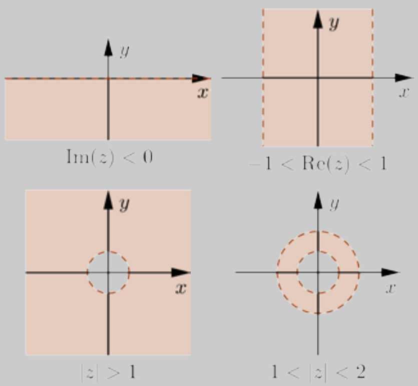
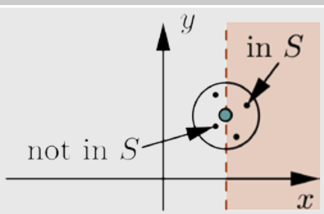
22		BY NC SA	Shih-Yuan Chen
22		BY NC SA	Shih-Yuan Chen
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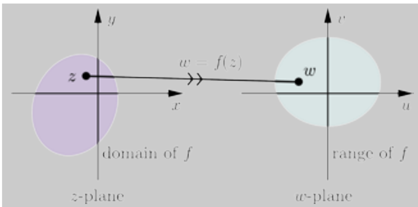
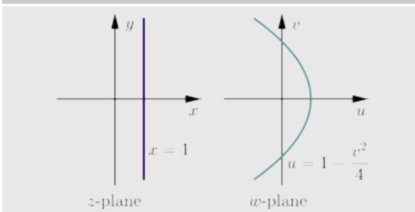
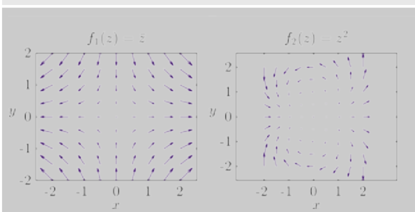
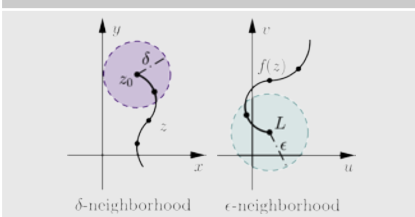
27		CC BY NC SA	Shih-Yuan Chen
28		CC BY NC SA	Shih-Yuan Chen
30		CC BY NC SA	Shih-Yuan Chen
33		CC BY NC SA	Shih-Yuan Chen

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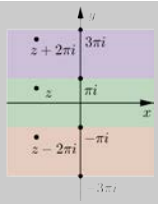
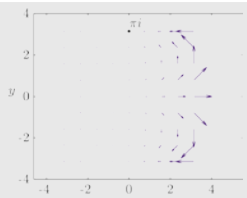
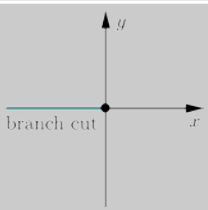
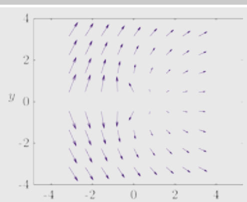
58		CC BY NC SA	Shih-Yuan Chen
59		CC BY NC SA	Shih-Yuan Chen
68		CC BY NC SA	Shih-Yuan Chen
69		CC BY NC SA	Shih-Yuan Chen