

工程數學--微分方程

Differential Equations (DE)

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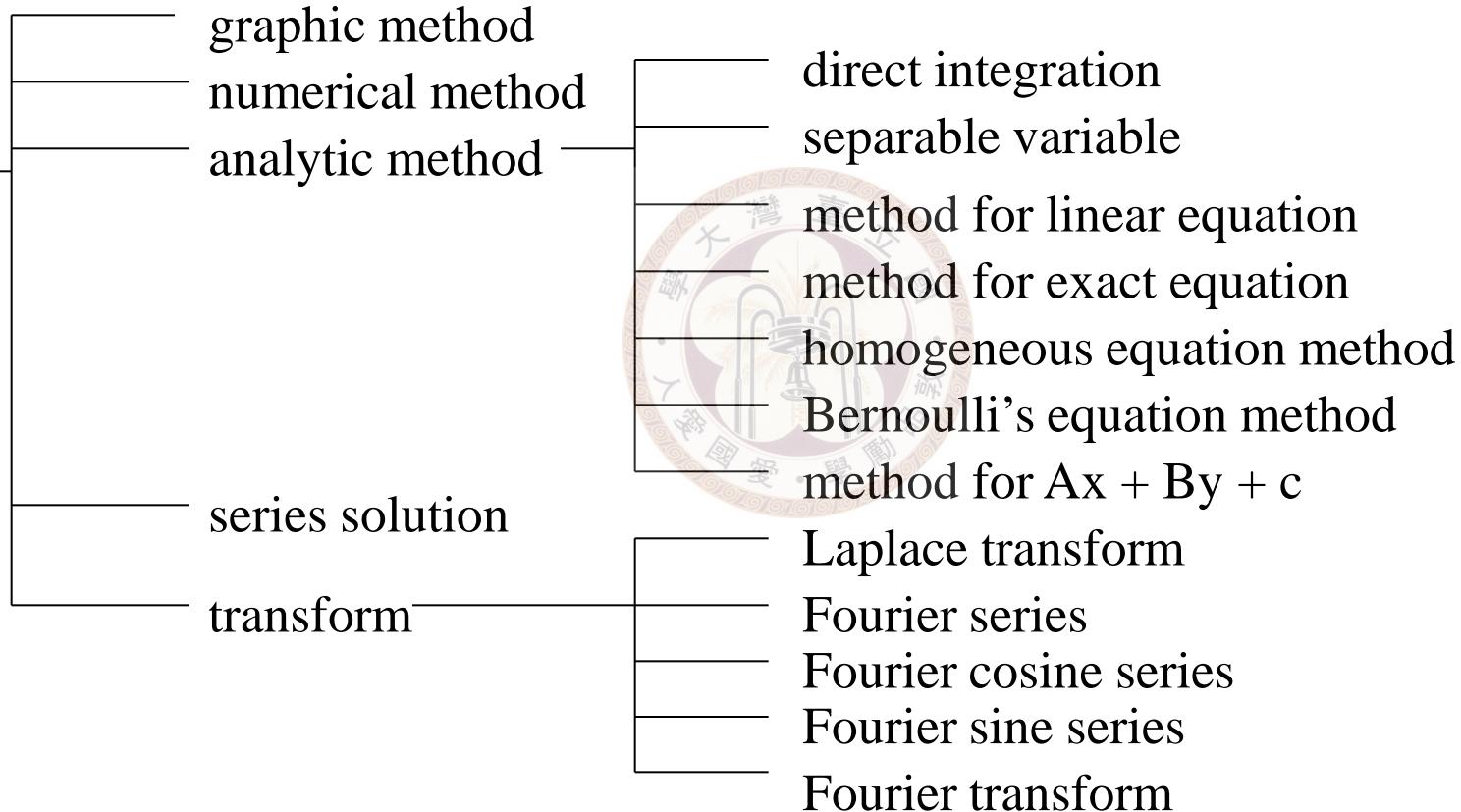
教學網頁：<http://djj.ee.ntu.edu.tw/DE.htm>



【本著作除另有註明外，採取創用CC「姓名標示－非商業性－相同方式分享」台灣3.0版授權釋出】

附錄一 Methods of Solving the First Order Differential Equation

29



Simplest method for solving the 1st order DE:

Direct Integration

$$dy(x)/dx = f(x)$$

$$\begin{aligned}y(x) &= \int f(x)dx \\&= F(x) + c\end{aligned}$$



where

$$\boxed{\frac{dF(x)}{dx} = f(x)}$$

Table of Integration

| | |
|------------------------------|--|
| $1/x$ | $\ln x + c$ |
| $\cos(x)$ | $\sin(x) + c$ |
| $\sin(x)$ | $-\cos(x) + c$ |
| $\tan(x)$ | $\ln \sec(x) + c$ |
| $\cot(x)$ | $\ln \sin(x) + c$ |
| a^x | $a^x/\ln(a) + c$ |
| $\frac{1}{x^2 + a^2}$ | $\frac{1}{a} \tan^{-1} \frac{x}{a} + c$ |
| $\frac{1}{\sqrt{a^2 - x^2}}$ | $\sin^{-1} \frac{x}{a} + c$ |
| $x e^{ax}$ | $\frac{e^{ax}}{a} \left(x - \frac{1}{a} \right) + c$ |
| $x^2 e^{ax}$ | $\frac{e^{ax}}{a} \left(x^2 - \frac{2x}{a} + \frac{2}{a^2} \right) + c$ |

2-2 Separable Variables

2-2-1 方法的限制條件

1st order DE 的一般型態: $dy(x)/dx = f(x, y)$

[Definition 2.2.1] (text page 45)

If $dy(x)/dx = f(x, y)$ and $f(x, y)$ can be separate as

$$f(x, y) = g(x)h(y)$$

i.e., $dy(x)/dx = g(x)h(y)$

then the 1st order DE is **separable** (or have separable variable).

條件： $dy(x)/dx = g(x)h(y)$

$$\frac{dy}{dx} = \cos(x)e^{x+2y}$$

$$\frac{dy}{dx} = x + y$$



2-2-2 解法

34

If $\frac{dy}{dx} = g(x)h(y)$, then

$$\begin{array}{c} \downarrow \\ \text{Step 1} \quad \frac{dy}{h(y)} = g(x)dx \\ \downarrow \\ p(y)dy = g(x)dx \end{array}$$

$$\begin{array}{c} \text{Step 2} \quad \underline{\int p(y)dy} = \underline{\int g(x)dx} \quad \text{個別積分} \\ \downarrow \quad \downarrow \\ \underline{P(y) + c_1} = \underline{G(x) + c_2} \\ \downarrow \\ P(y) = G(x) + c \end{array}$$



分離變數

where

$$p(y) = 1/h(y)$$

where

$$\frac{dP(y)}{dy} = p(y)$$

$$\frac{dG(x)}{dx} = g(x)$$

Extra Step: (a) Initial conditions

(b) Check the singular solution (i.e., the constant solution)

Extra Step (b) Check the singular solution:

Suppose that y is a constant r

$$\frac{dy}{dx} = g(x)h(y)$$



$$0 = g(x)h(r)$$



$$h(r) = 0$$



solution for r



See whether the solution is a special case of the general solution.



2-2-3 Examples

Example 1 (text page 46)

$$\begin{aligned}
 & (1+x) dy - y dx = 0 \quad \xrightarrow{\quad} \frac{dy}{dx} = \frac{y}{1+x} \\
 \text{Step 1} \quad & \frac{dy}{y} = \frac{dx}{1+x} \\
 & \downarrow \qquad \downarrow \\
 \text{Step 2} \quad & \ln|y| = \ln|1+x| + c_1 \\
 & \downarrow \\
 & |y| = e^{\ln|1+x|} e^{c_1} \longrightarrow y = \pm e^{c_1} e^{\ln|1+x|} \\
 & y = \pm e^{c_1} |1+x| = \pm e^{c_1} (1+x) \\
 & \downarrow \\
 & y = c(1+x) \quad c = \pm e^{c_1}
 \end{aligned}$$



Extra Step (b)
check the singular
solution

$$\text{set } y = r,$$

$$0 = r/(1+x)$$

$$r = 0,$$

$$y = 0$$

(a special case of the
general solution)

Example 練習小技巧

遮住解答和筆記，自行重新算一次

(任何和解題有關的提示皆遮住)



Exercise 練習小技巧

初學者，先針對有解答的題目作練習

累積一定的程度和經驗後，再多練習沒有解答的題目

將題目依類型分類，多練習解題正確率較低的題型

動筆自己算，就對了

Example 2 (with **initial condition** and **implicit solution**, text page 46)

$$\frac{dy}{dx} = -\frac{x}{y},$$

$$y(4) = -3$$

Step 1 $y dy = -x dx$

Step 2 $y^2 / 2 = -x^2 / 2 + c$

$$x^2 + y^2 = 25 \quad (\text{implicit solution})$$

$$y = \sqrt{25 - x^2} \quad \text{invalid}$$

$$y = -\sqrt{25 - x^2} \quad \text{valid}$$

(explicit solution)

Extra Step (b)

check the singular solution

Extra Step (a)

$$4.5 = -8 + c, \quad c = 12.5$$

Example 3 (with singular solution, text page 47)

Step 1

$$\frac{dy}{dx} = y^2 - 4$$

$$\frac{dy}{y^2 - 4} = dx$$

Step 2

$$\frac{1}{4} \frac{dy}{y-2} - \frac{1}{4} \frac{dy}{y+2} = dx$$

$$\frac{1}{4} \ln|y-2| - \frac{1}{4} \ln|y+2| = x + c_1$$

$$\ln \left| \frac{y-2}{y+2} \right| = 4x + 4c_1$$

$$\frac{y-2}{y+2} = \pm e^{4x+4c_1} = ce^{4x}$$

$$c = \pm e^{4c_1}$$



Extra Step (b)

check the singular solution

$$\frac{dy}{dx} = y^2 - 4$$

set $y = r$,

$$0 = r^2 - 4$$

$$r = \pm 2,$$

$$y = \pm 2$$

or $y = \pm 2$

Example 4 (text page 47)

自修

注意如何計算 $\int \frac{\sin(2x)}{\cos x} dx, \int ye^{-y} dy$



Example in the bottom of page 48

$$\frac{dy}{dx} = xy^{1/2}, \quad y(0) = 0$$



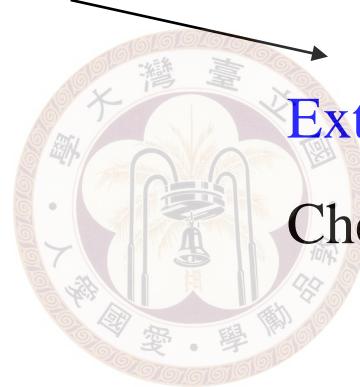
Step 1



Step 2



Extra Step (a)



Extra Step (b)

Check the singular solution

Solution: $y = \frac{1}{16}x^4$ or $y = 0$ 其實，還有更多的解

$$\frac{dy}{dx} = xy^{1/2}, \quad y(0) = 0$$

solutions: (1) $y = \frac{1}{16}x^4$ (2) $y = 0$

$$(3) \quad y = \begin{cases} \frac{1}{16}(x^2 - b^2)^2 & \text{for } x \leq b \\ 0 & \text{for } b < x < a \\ \frac{1}{16}(x^2 - a^2)^2 & \text{for } x \geq a \end{cases} \quad b \leq 0 \leq a$$

2-2-4 IVP 是否有唯一解？

$$\frac{dy}{dx} = f(x, y) \quad y(x_0) = y_0$$

這個問題有唯一解的條件：(Theorem 1.2.1, text page 15)

如果 $f(x, y), \frac{\partial}{\partial y} f(x, y)$ 在 $x = x_0, y = y_0$ 的地方為 continuous

則必定存在一個 h ，使得 IVP 在 $x_0 - h < x < x_0 + h$ 的區間當中有唯一解

2-2-5 Solutions Defined by Integral

$$(1) \quad \frac{d}{dx} \int_{x_0}^x g(t) dt = g(x)$$

(2) If $dy/dx = g(x)$ and $y(x_0) = y_0$, then

$$y(x) = y_0 + \int_{x_0}^x g(t) dt$$

積分 (integral, antiderivative) 難以計算的 function ,

被稱作是 nonelementary

如 e^{-x^2} , $\sin x^2$

此時, solution 就可以寫成 $y(x) = y_0 + \int_{x_0}^x g(t) dt$ 的型態

Example 5 $\frac{dy}{dx} = e^{-x^2}$ $y(3) = 5$

Solution $y(x) = 5 + \int_3^x e^{-t^2} dt$

或者可以表示成 complementary error function

$$y(x) = 5 + \frac{\sqrt{\pi}}{2} (\operatorname{erfc}(3) - \operatorname{erfc}(x))$$

- error function (useful in probability)

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

- complementary error function

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt = 1 - \operatorname{erf}(x)$$

See text page 59 in Section 2.3

2-2-6 本節要注意的地方

(1) 複習並背熟幾個重要公式的積分

(2) 別忘了加 c

並且熟悉什麼情況下 c 可以合併和簡化

(3) 若時間允許，別忘了計算 singular solution

(4) 多練習，加快運算速度



附錄二 微分方程查詢

<http://integrals.wolfram.com/index.jsp>

輸入數學式，就可以查到積分的結果

範例：

- 先到integrals.wolfram.com/index.jsp 這個網站
- 在右方的空格中輸入數學式，例如

數學式

Wolfram Mathematica[®]
ONLINE INTEGRATOR
The world's only full-power integration solver

HOW TO ENTER INPUT | RANDOM EXAMPLE

$\int \cos(ax)+b \, dx$

Compute Online With Mathematica

(c) 接著按 “Compute Online with Mathematica”

就可以算出積分的結果

The screenshot shows the Wolfram Mathematica Online Integrator interface. At the top, it says "Wolfram Mathematica ONLINE INTEGRATOR" and "The world's only full-power integration solver". Below that is a search bar containing the integral expression $\int \cos(ax)+b dx$. A red arrow labeled "按" points to the "Compute Online With Mathematica" button. The result section below shows the integral $\int b + \cos(ax) dx =$ followed by the antiderivative $b x + \frac{\sin(ax)}{a}$. A red arrow labeled "結果" points to this result area. At the bottom, it says "Time to compute: < 0.01 second".

(d) 有時，對於一些較複雜的數學式，下方還有連結，點進去就可 50
以看到相關的解說

Wolfram Mathematica®
ONLINE INTEGRATOR
The world's only full-power integration solver

HOW TO ENTER INPUT | RANDOM EXAMPLE

$\int \exp(-a*x^2) dx$

Compute Online With Mathematica

Traditional Form | Input Form | Output Form

$$\int e^{-ax^2} dx = \frac{\sqrt{\pi} \operatorname{erf}(\sqrt{a} x)}{2 \sqrt{a}}$$

Time to compute: < 0.01 second

erf(x): Erf[x]: error function [properties]

連結

其他有用的網站

<http://mathworld.wolfram.com/>

對微分方程的定理和名詞作介紹的百科網站

<http://www.sosmath.com/tables/tables.html>

眾多數學式的 mathematical table (不限於微分方程)

<http://www.seminaire-sherbrooke.qc.ca/math/Pierre/Tables.pdf>

眾多數學式的 mathematical table , 包括 convolution, Fourier transform, Laplace transform, Z transform

軟體當中 , Maple, Mathematica, Matlab 皆有微積分結果查詢有功能

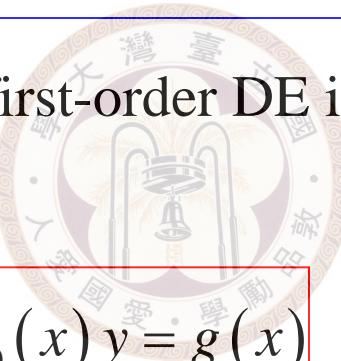
2-3 Linear Equations

“friendly” form of DEs

2-3-1 方法的適用條件

[Definition 2.3.1] The first-order DE is a **linear equation** if it has the following form:

$$a_1(x) \frac{dy}{dx} + a_0(x)y = g(x)$$



$g(x) = 0$: **homogeneous**

$g(x) \neq 0$: **nonhomogeneous**

Standard form:

$$\frac{dy}{dx} + P(x)y = f(x)$$

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \xrightarrow{\text{Divide by } a_1(x)} \frac{dy}{d} + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)}$$

許多自然界的現象，皆可以表示成 linear first order DE

2-3-2 解法的推導

$$\frac{dy}{dx} + P(x)y = f(x)$$

子問題 1

$$\frac{dy_c}{dx} + P(x)y_c = 0$$

Find the **general** solution $y_c(x)$

(homogeneous solution)

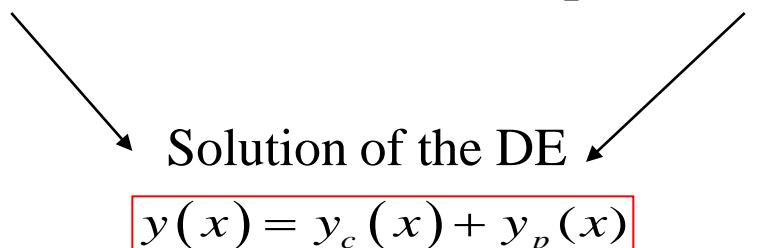
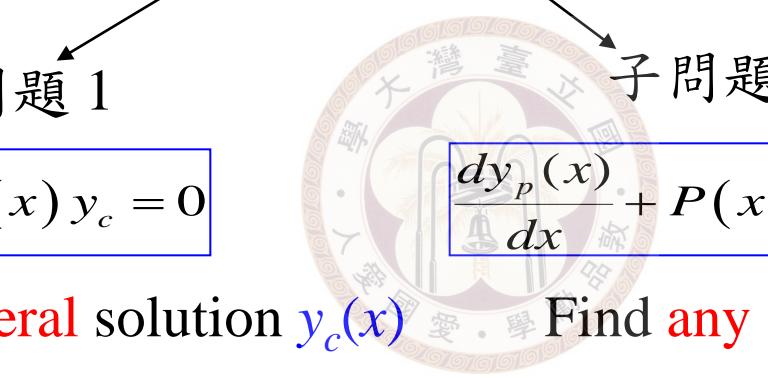
子問題 2

$$\frac{dy_p(x)}{dx} + P(x)y_p(x) = f(x)$$

Find **any** solution $y_p(x)$

(particular solution)

$$y(x) = y_c(x) + y_p(x)$$



- $y_c + y_p$ is a solution of the linear first order DE, since

$$\begin{aligned}
 & \frac{d(y_c + y_p)}{dx} + P(x)(y_c + y_p) \\
 &= \left(\frac{dy_c}{dx} + P(x)y_c \right) + \left(\frac{dy_p}{dx} + P(x)y_p \right) \\
 &= 0 + f(x) = f(x)
 \end{aligned}$$

- Any solution of the linear first order DE should have the form $y_c + y_p$.

The proof is as follows. If y is a solution of the DE, then

$$\begin{aligned}
 & \frac{dy}{dx} + P(x)y - \left(\frac{dy_p}{dx} + P(x)y_p \right) = f(x) - f(x) = 0 \\
 & \frac{d(y - y_p)}{dx} + P(x)(y - y_p) = 0
 \end{aligned}$$

Thus, $y - y_p$ should be the solution of $\frac{dy_c}{dx} + P(x)y_c = 0$

y should have the form of $y = y_c + y_p$

Solving the homogeneous solution $y_c(x)$ (子問題一)

$$\boxed{\frac{dy_c}{dx} + P(x)y_c = 0}$$

↓ separable variable

$$\frac{dy_c}{y_c} = -P(x)dx$$

$$\downarrow \qquad \qquad \qquad \downarrow$$

$$\ln|y_c| = \int -P(x)dx + c_1$$

↓

$$\boxed{y_c = ce^{-\int P(x)dx}}$$



Set $y_1 = e^{-\int P(x)dx}$, then $y_c = cy_1$

Solving the particular solution $y_p(x)$ (子問題二)

$$\boxed{\frac{dy_p(x)}{dx} + P(x)y_p(x) = f(x)}$$

Set $y_p(x) = u(x)y_1(x)$ (猜測 particular solution 和 homogeneous solution 有類似的關係)

$$u(x)\frac{dy_1(x)}{dx} + y_1(x)\frac{du(x)}{dx} + P(x)u(x)y_1(x) = f(x)$$

$$y_1(x)\frac{du(x)}{dx} + u(x)\left[\frac{dy_1(x)}{dx} + P(x)y_1(x)\right] = f(x)$$

equal to zero

$$y_1(x)\frac{du(x)}{dx} = f(x)$$

$$du(x) = \frac{f(x)}{y_1(x)}dx \longrightarrow u(x) = \int \frac{f(x)}{y_1(x)}dx \longrightarrow \boxed{y_p(x) = y_1(x) \int \frac{f(x)}{y_1(x)}dx}$$

$$y_c = ce^{-\int P(x)dx}$$

$$y_p(x) = e^{-\int P(x)dx} \int [e^{\int P(x)dx} f(x)]dx$$

solution of the linear 1st order DE:

$$y(x) = c e^{-\int P(x)dx} + e^{-\int P(x)dx} \int [e^{\int P(x)dx} f(x)]dx$$

where c is any constant

$e^{\int P(x)dx}$: integrating factor

2-3-3 解法

(Step 1) Obtain the **standard form** and find $P(x)$

(Step 2) Calculate $e^{\int P(x)dx}$

(Step 3) The standard form of the linear 1st order DE can be rewritten as:

$$\frac{d}{dx} \left[e^{\int P(x)dx} y \right] = e^{\int P(x)dx} f(x) \quad \text{remember it}$$

(Step 4) Integrate both sides of the above equation

$$e^{\int P(x)dx} y = \int e^{\int P(x)dx} f(x) dx + c,$$

$$y = e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x) dx + c e^{-\int P(x)dx}$$

(Extra Step) (a) Initial value

(c) Check the Singular Point

or remember it,
skip Step 3

$$a_1(x) \frac{dy}{dx} + a_0(x)y = g(x) \quad \frac{dy}{dx} + P(x)y = f(x)$$

Singular points: the locations where $a_1(x) = 0$

i.e., $P(x) \rightarrow \infty$

More generally, even if $a_1(x) \neq 0$ but $P(x) \rightarrow \infty$ or $f(x) \rightarrow \infty$, then the location is also treated as a singular point.

- (a) **Sometimes, the solution may not be defined on the interval including the singular points.** (such as Example 4)
- (b) Sometimes the solution can be defined at the singular points, such as Example 3

More generally, even if $a_1(x) \neq 0$ but $P(x) \rightarrow \infty$ or $f(x) \rightarrow \infty$, then the location is also treated as a singular point.

Exercise 29

$$(x+1)\frac{dy}{dx} + y = \ln|x|$$



2-3-4 例子

Example 2 (text page 55)

$$\frac{dy}{dx} - 3y = 6$$

Step 1

$$P(x) = -3$$

Step 2

$$e^{\int P(x) dx} = e^{-3x}$$

Step 3

$$\frac{d}{dx} [e^{-3x} y] = 6e^{-3x}$$

Step 4

$$e^{-3x} y = -2e^{-3x} + c$$

$$y = -2 + ce^{3x}$$

Extra Step (c)
check the singular point



為何在此時可以將
 $-3x+c$ 簡化成 $-3x$?

或著，跳過 Step 3，直接代公式

$$y = e^{-\int P(x) dx} \int e^{\int P(x) dx} f(x) dx + ce^{-\int P(x) dx}$$

Example 3 (text page 56)

$$x \frac{dy}{dx} - 4y = x^6 e^x$$

Step 1 $\frac{dy}{dx} - 4\frac{y}{x} = x^5 e^x, P(x) = -\frac{4}{x}$

Step 2 $e^{\int P(x)dx} = e^{-4\ln|x|} = |x|^{-4}$
若只考慮 $x > 0$ 的情形, $e^{\int P(x)dx} = x^{-4}$

Step 3 $\frac{d}{dx}[x^{-4}y] = xe^x$

Step 4 $x^{-4}y = (x-1)e^x + c$

$$y = (x^5 - x^4)e^x + cx^4$$

x 的範圍: $(0, \infty)$

Extra Step (c)

check the singular point

$$x = 0$$

思考 : $x < 0$ 的情形

Example 4 (text page 57)

$$(x^2 - 9) \frac{dy}{dx} + xy = 0$$

$$\frac{dy}{dx} + \frac{x}{x^2 - 9} y = 0$$

$$P(x) = \frac{x}{x^2 - 9}$$

$$e^{\int \frac{x}{x^2 - 9} dx} = e^{\frac{1}{2} \ln|x^2 - 9|} = \sqrt{|x^2 - 9|}$$

$$\frac{d}{dx} \sqrt{|x^2 - 9|} \cdot y = 0$$

$$\sqrt{|x^2 - 9|} \cdot y = c$$

$$y = \frac{c}{\sqrt{|x^2 - 9|}}$$

Extra Step (c)
check the singular point



defined for $x \in (-\infty, -3), (-3, 3), (3, \infty)$

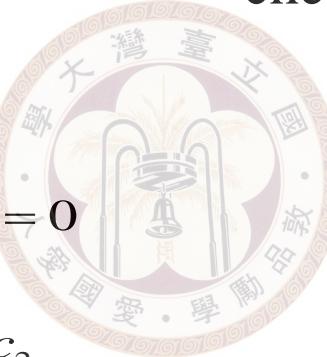
not includes the points of $x = -3, 3$

Example 6 (text, page 58)

$$\boxed{\frac{dy}{dx} + y = f(x)}$$

$y(0) = 0 \quad f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$

check the singular point



$0 \leq x \leq 1$
 $\frac{d}{dx}(e^x y) = e^x$
 $e^x y = e^x + c_1$
 $y = 1 + c_1 e^{-x}$

from initial condition

$x > 1$
 $\frac{d}{dx}(e^x y) = 0$
 $e^x y = c_2$
 $y = c_2 e^{-x}$

要求 $y(x)$ 在 $x = 1$ 的地方
為 continuous

$y = 1 - e^{-x}$

$y = (e - 1)e^{-x}$

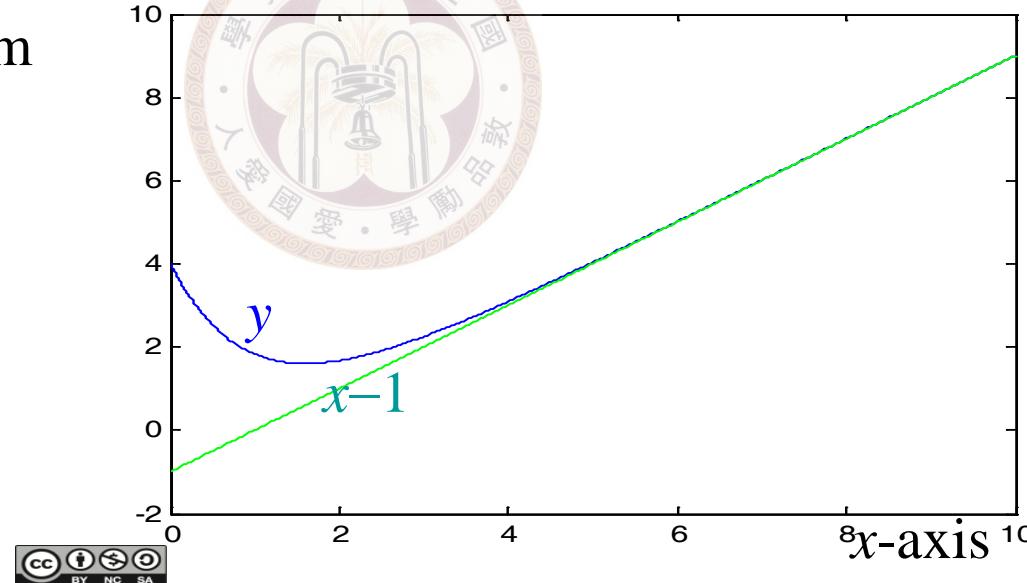
2-3-5 名詞和定義

(1) transient term, stable term

Example 5 (text page 58) 的解為 $y = x - 1 + 5e^{-x}$

$5e^{-x}$: transient term 當 x 很大時會消失

$x - 1$: stable term



(2) piecewise continuous

A function $g(x)$ is piecewise continuous in the region of $[x_1, x_2]$ if $g'(x)$ exists for any $x \in [x_1, x_2]$.

In Example 6, $f(x)$ is piecewise continuous in the region of $[0, 1)$ or $(1, \infty)$

(3) Integral (積分) 有時又被稱作 antiderivative

$$(4) \text{ error function} \quad \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$\text{complementary error function} \quad \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt = 1 - \text{erf}(x)$$

(5) sine integral function

$$\text{Si}(x) = \int_0^x \frac{\sin(t)}{t} dt$$

Fresnel integral function

$$S(x) = \int_0^x \sin(\pi t^2 / 2) dt$$

$$(6) \quad \frac{dy}{dx} + P(x)y = f(x)$$

$f(x)$ 常被稱作 input 或 deriving function

Solution $y(x)$ 常被稱作 output 或 response



2-3-6 小技巧

When $\frac{dy}{dx}$ is not easy to calculate:

Try to calculate $\frac{dx}{dy}$

Example: $\frac{dy}{dx} = \frac{1}{x + y^2}$ (not linear, not separable)

$$\downarrow$$

$$\frac{dx}{dy} = x + y^2 \quad (\text{linear})$$

$$\downarrow$$

$$x = -y^2 - 2y - 2 + ce^{-y} \quad (\text{implicit solution})$$



2-3-7 本節要注意的地方

- (1) 要先將 linear 1st order DE 變成 standard form
- (2) 別忘了 singular point

注意：singular point 和 Section 2-2 提到的 singular solution 不同

- (3) 記熟公式

$$\frac{d}{dx} \left[e^{\int P(x)dx} y \right] = e^{\int P(x)dx} f(x)$$

或

$$y = e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x) dx + ce^{-\int P(x)dx}$$

- (4) 計算時， $e^{\int P(x)dx}$ 的常數項可以忽略

太多公式和算法，怎麼辦？

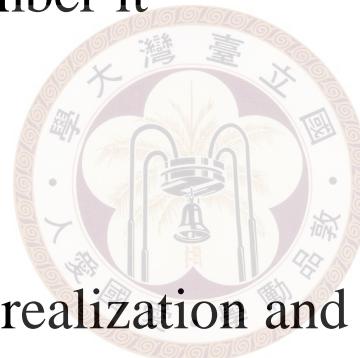
最上策： realize + remember it

上策： realize it

中策： remember it

下策： read it without realization and remembrance

最下策： rest z.....z.....z.....

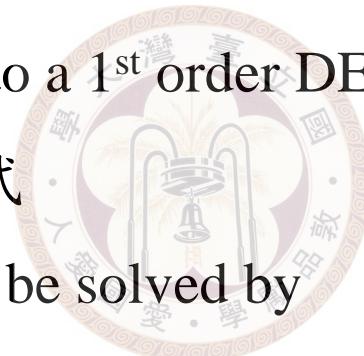


Chapter 3 Modeling with First-Order Differential Equations

應用題

(1) Convert a question into a 1st order DE.

將問題翻譯成數學式



(2) Many of the DEs can be solved by

Separable variable method or

Linear equation method

(with integration table remembrance)

3-1 Linear Models

Growth and Decay (Examples 1~3)

Change the Temperature (Example 4)

Mixtures (Example 5)

Series Circuit (Example 6)

可以用 Section 2-3 的方法來解



Example 1 (an example of growth and decay, text page 83)

Initial: A culture (培養皿) initially has P_0 number of bacteria.

$$\text{翻譯} \rightarrow A(0) = P_0$$

The other initial condition: At $t = 1$ h, the number of bacteria is measured to be $3P_0/2$.

$$\text{翻譯} \rightarrow A(1) = 3P_0/2$$

關鍵句: If the **rate of growth** is proportional to the number of bacteria $A(t)$ presented at time t ,

$$\text{翻譯} \rightarrow \frac{dA}{dt} = kA \quad k \text{ is a constant}$$

Question: determine the time necessary for the number of bacteria to triple
 翻譯 → find t such that $A(t) = 3P_0$

這裡將課本的 $P(t)$ 改成 $A(t)$

$$\frac{dA}{dt} = kA$$

$A(0) = P_0, A(1) = 3P_0/2$ 可以用什麼方法解？

Step 1 $\frac{dA}{A} = kdt$

Step 2 $\ln|A| = kt + c_1$

$$|A| = e^{kt+c_1}$$

Extra

Step (a) (1) $P_0 = c \cdot 1$



$$c = P_0$$

(2) $3P_0 / 2 = ce^k$

$$k = \ln(3/2) = 0.4055$$

$$A = P_0 e^{0.4055t}$$



針對這一題的問題

$$3P_0 = P_0 e^{0.4055t}$$

$$t = \ln(3) / 0.4055 \approx 2.71h$$

課本用 linear (Section 2.3) 的方法來解 Example 1

思考：為什麼此時需要兩個 initial values 才可以算出唯一解？



Example 4 (an example of temperature change, text page 85)

Initial: When a cake is removed from an oven, its temperature is measured at 300° F. 翻譯 → $T(0) = 300$

The other initial condition: Three minutes later its temperature is 200° F.
翻譯 → $T(3) = 200$

question: Suppose that the room temperature is 70° F. How long will it take for the cake to cool off to 75° F? (註：這裡將課本的問題做一些修改)

翻譯 → find t such that $T(t) = 75$.

另外，根據題意，了解這是一個物體溫度和周圍環境的溫度交互作用的問題，所以 $T(t)$ 所對應的 DE 可以寫成

$$\boxed{\frac{dT}{dt} = k(T - 70)}$$

k is a constant

$$\frac{dT}{dt} = k(T - 70)$$

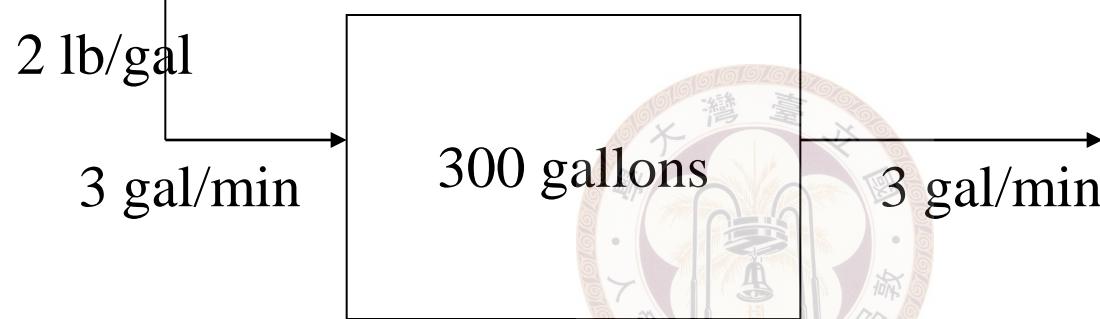
$$T(0) = 300 \quad T(3) = 200$$

課本用 separable variable 的方法解
如何用 linear 的方法來解？



Example 5 (an example for mixture, text page 86)

Concentration:

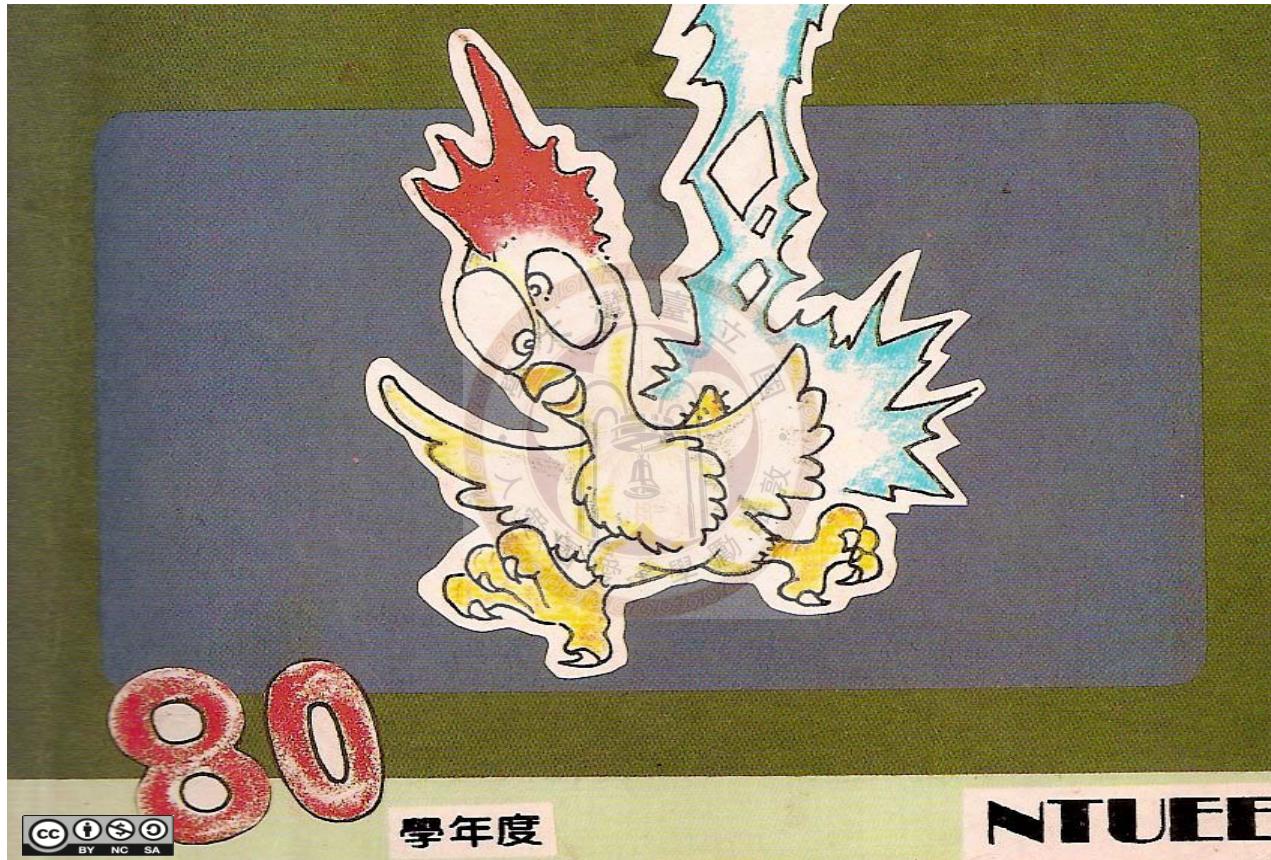


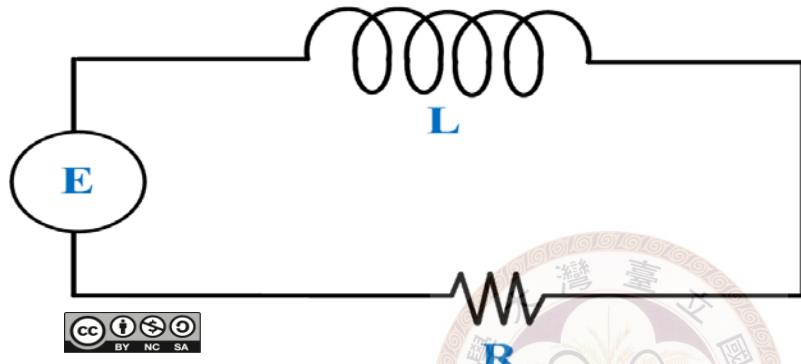
A: the amount of salt in the tank

$$\frac{dA}{dt} = (\text{input rate of salt}) - (\text{output rate of salt})$$

$$= 3 \cdot 2 - \frac{3A}{300}$$

80

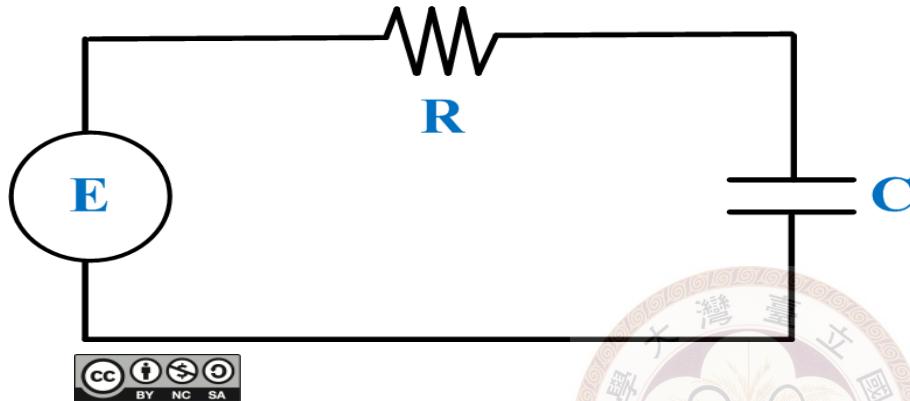




LR series circuit

From Kirchhoff's second law

$$L \frac{di}{dt} + Ri = E(t)$$

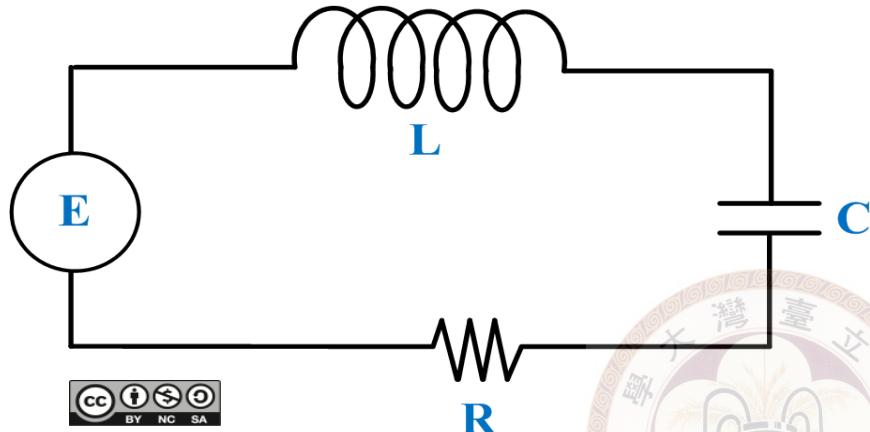


RC series circuit

$$\frac{q}{C} + Ri = E(t)$$

q: 電荷

$$\frac{q}{C} + R \frac{dq}{dt} = E(t)$$

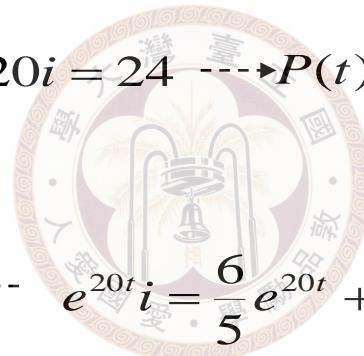


How about an *LRC* series circuit?

$$\frac{q}{C} + R \frac{dq}{dt} + L \frac{d^2q}{dt^2} = E(t)$$

Example 6 (text page 88) LR series circuit

- $E(t)$: 12 volt, • inductance: $1/2$ henry,
- resistance: 10 ohms, • initial current: 0



$$\boxed{\frac{1}{2} \frac{di}{dt} + 10i = 12} \rightarrow \frac{di}{dt} + 20i = 24 \rightarrow P(t) = 20 \rightarrow e^{\int P(t)dt} = e^{20t+c_1}$$

這裡 $+ c_1$ 可省略

$$\boxed{i(t) = \frac{6}{5} + ce^{-20t}}$$

$$e^{20t}i = \frac{6}{5}e^{20t} + c \leftarrow \frac{d}{dt}e^{20t}i = 24e^{20t}$$

$$i(0) = 0 \rightarrow 0 = \frac{6}{5} + c \rightarrow c = -\frac{6}{5}$$

$$\boxed{i(t) = \frac{6}{5} - \frac{6}{5}e^{-20t}}$$

Circuit problem for t is small and $t \longrightarrow \infty$

For the LR circuit: L R
transient stable

For the RC circuit: R C
transient stable



3-2 Nonlinear Models

可以用 separable variable 或其他的方法來解

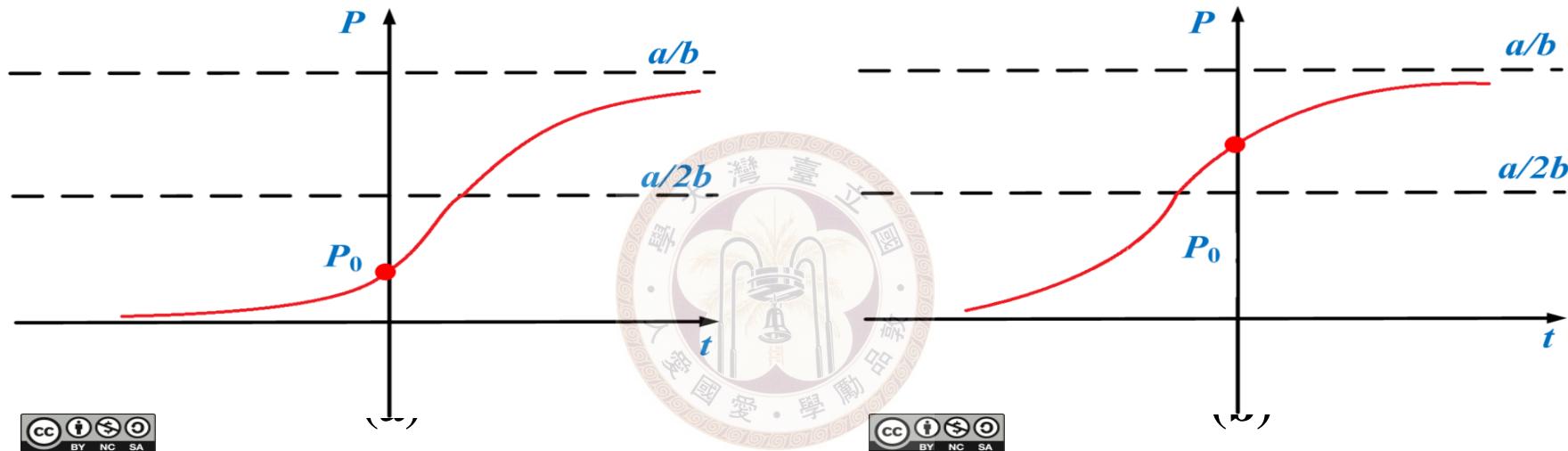
3-2-1 Logistic Equation

used for describing the growth of population

$$\frac{dP}{dt} = P(a - bP) = bP\left(\frac{a}{b} - P\right)$$

The solution of a **logistic equation** is called the **logistic function**.

Two stable conditions: $P = 0$ and $P = \frac{a}{b}$.



Logistic curves for differential initial conditions

Solving the logistic equation

$$\frac{dP}{dt} = P(a - bP)$$



$$\frac{dP}{P(a - bP)} = dt$$

separable
variable

$$\left(\frac{1/a}{P} + \frac{b/a}{a - bP} \right) dP = dt$$

$$\frac{1}{a} \ln|P| - \frac{1}{a} \ln|a - bP| = t + c$$

$$\ln \left| \frac{P}{a - bP} \right| = at + ac$$

$$\frac{P}{a - bP} = c_1 e^{at}$$

$$c_1 = \pm e^{ac}$$



註 :

$$\int \frac{-b}{a - bP} dP = \int \frac{\frac{d}{dP}(a - bP)}{a - bP} dP = \ln|a - bP| + c_0$$

(with initial condition $P(0) = P_0$)

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}$$

logistic function

Example 1 (text page 96) There are 1000 students.

- Suppose a student carrying a flu virus returns to an isolate college campus of 1000 students.

翻譯 → $x(0) = 1$

- If it is assumed that the rate at which the virus spreads is proportional not only to the number x of infected students but also to the number of students not infected,

翻譯 →
$$\frac{dx(t)}{dt} = kx(1000 - x) \quad k \text{ is a constant}$$

- determine the number of infected students after 6 days
翻譯 → find $x(6)$

- if it is further observed that after 4 days $x(4) = 50$

整個問題翻譯成

$$\frac{dx(t)}{dt} = kx(1000 - x)$$

Initial: $x(0) = 1, x(4) = 50$

find $x(6)$

可以用 separable variable 的方法



$$\frac{dx(t)}{dt} = kx(1000 - x)$$

$$\frac{dx(t)}{x(1000 - x)} = kdt$$

$$\frac{1}{1000} \left(\frac{dx}{x} + \frac{dx}{1000 - x} \right) = kdt$$

$$\frac{dx}{x} - \frac{dx}{x - 1000} = 1000kdt$$

$$\ln|x| - \ln|x - 1000| = 1000kt + c_1$$

$$\left| \frac{x}{x - 1000} \right| = e^{1000kt + c_1}$$

$$\frac{x}{x - 1000} = c_2 e^{1000kt} \quad (c_2 = \pm e^{c_1})$$

$$(c_2 e^{1000kt} - 1)x = c_2 1000e^{1000kt}$$

$$x = \frac{1000}{1 - c e^{-1000kt}} \quad (c = c_2^{-1})$$

$$1 = \frac{1000}{1 - c}$$

$$c = -999$$

$$x = \frac{1000}{1 + 999 e^{-1000kt}}$$

$$50 = \frac{1000}{1 + 999 e^{-4000k}}$$

$$-1000k = -0.9906$$

$$x = \frac{1000}{1 + 999 e^{-0.9906t}}$$

$$x(0) = 1$$

$$x(4) = 50$$

$$x(6) \approx 276$$

Logistic equation 的變形

$$(1) \quad \frac{dP}{dt} = P(a - bP) \pm h \quad \text{人口有遷移的情形}$$

$$(2) \quad \frac{dP}{dt} = P(a - bP) - cP \quad \text{遷出的人口和人口量呈正比}$$

$$(3) \quad \frac{dP}{dt} = P(a - bP) + ce^{-kP} \quad \text{人口越多，遷入的人口越少}$$

$$(4) \quad \frac{dP}{dt} = P(a - b \ln P)$$

$$= bP(a/b - \ln P)$$

飽合人口為 $e^{a/b}$
 $\ln \frac{\text{飽合人口}}{P}$
 人口增加量，和
 呈正比

3-2-2 化學反應的速度



- Use compounds A and B to form compound C
- $x(t)$: the amount of C
- To form a unit of C requires s_1 units of A and s_2 units of B
- a : the original amount of A
- b : the original amount of B
- The rate of generating C is proportional to the product of the amount of A and the amount of B



$$\frac{dx(t)}{dt} = k(a - s_1x)(b - s_2x)$$

See Example 2

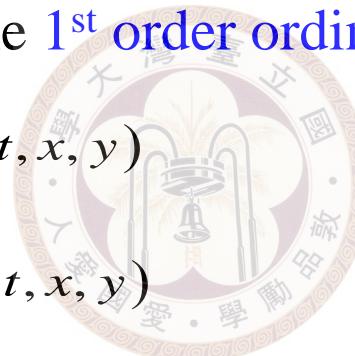
3-3 Modeling with Systems of DEs

Some Systems are hard to model by one dependent variable

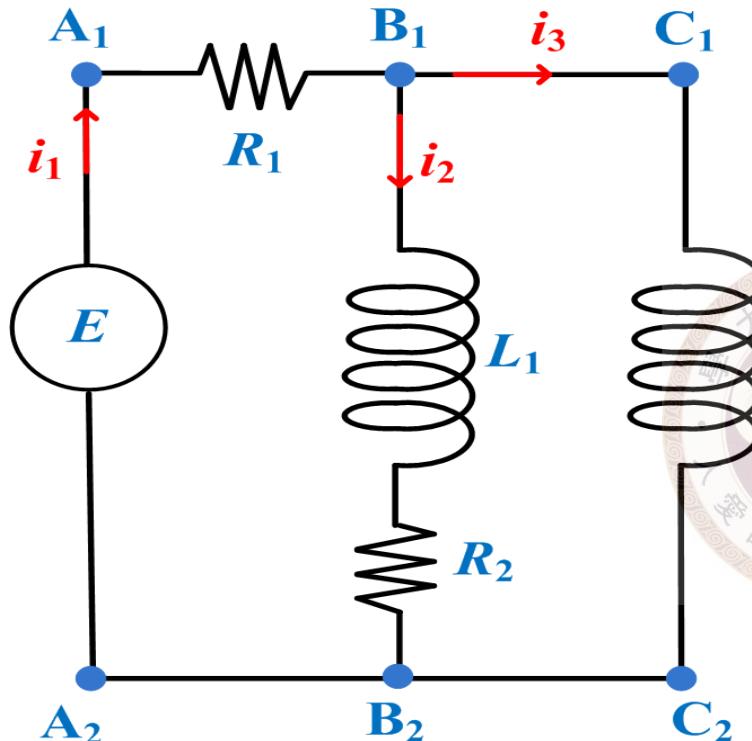
but can be modeled by the 1st order ordinary differential equation

$$\frac{dx(t)}{dt} = g_1(t, x, y)$$

$$\frac{dy(t)}{dt} = g_2(t, x, y)$$



They should be solved by the Laplace Transform and other methods



from Kirchhoff's 1st law

$$i_1(t) = i_2(t) + i_3(t)$$

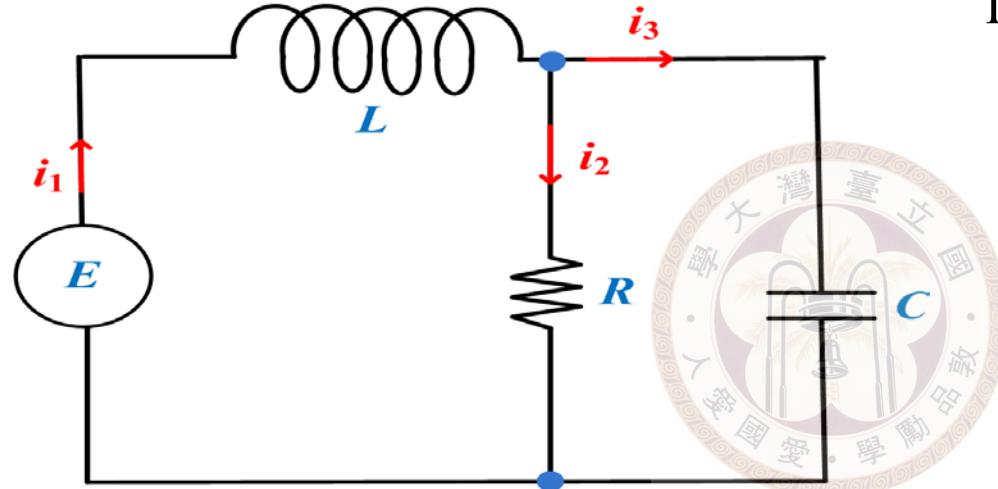
from Kirchhoff's 2nd law

$$(1) E(t) = i_1 R_1 + L_1 \frac{di_2(t)}{dt} + i_2 R_2$$

$$(2) E(t) = i_1 R_1 + L_2 \frac{di_3(t)}{dt}$$

Three dependent variable

We can only simplify it into two dependent variable



from Kirchhoff's 1st law

$$i_1(t) = i_2(t) + i_3(t)$$

from Kirchhoff's 2nd law

$$(1) \quad E(t) = L \frac{di_1(t)}{dt} + i_2(t)R$$

$$(2) \quad \frac{q_3(t)}{C} = i_2(t)R$$



$$\frac{1}{C} [i_1(t) - i_2(t)] = R \frac{d}{dt} i_2(t)$$

Chapter 3: 訓練大家將和 variation 有關的問題寫成 DE 的能力

..... the variation is proportional to



練習題

Section 2-2: 4, 7, 12, 13, 18, 21, 25, 28, 32, 46

Section 2-3: 7, 9, 13, 15, 21, 27, 29, 47, 49(a), 50(a)

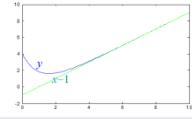
Section 3-1: 4, 5, 10, 15, 20, 29, 32

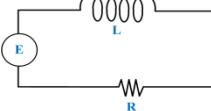
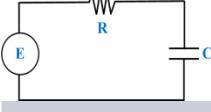
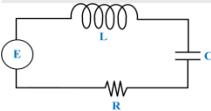
Section 3-2: 2, 5, 14, 15

Section 3-3: 12, 13

Review 3: 3, 4, 11, 12



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