

# 工程數學--微分方程

## Differential Equations (DE)

授課者：丁建均

教學網頁：<http://djj.ee.ntu.edu.tw/DE.htm>



【本著作除另有註明外，採取創用CC「姓名標示-非商業性-相同方式分享」台灣3.0版授權釋出】

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上課時間：星期三 第 3, 4 節 (AM 10:20~12:10)

星期五 第 2 節 (AM 9:10~10:00)

上課地點：電二143

課本：**"Differential Equations-with Boundary-Value Problem",  
7<sup>th</sup> edition, Dennis G. Zill and Michael R. Cullen**

評分方式：四次作業一次小考 10%，期中考 45%，期末考 45%

## 注意事項：

(1) 請上課前，來這個網頁，將上課資料印好。

<http://djj.ee.ntu.edu.tw/DE.htm>

(2) 請各位同學踴躍出席。

(3) 作業不可以抄襲。作業若寫錯但有用心寫仍可以有40%~90%的分數，但抄襲或借人抄襲不給分。

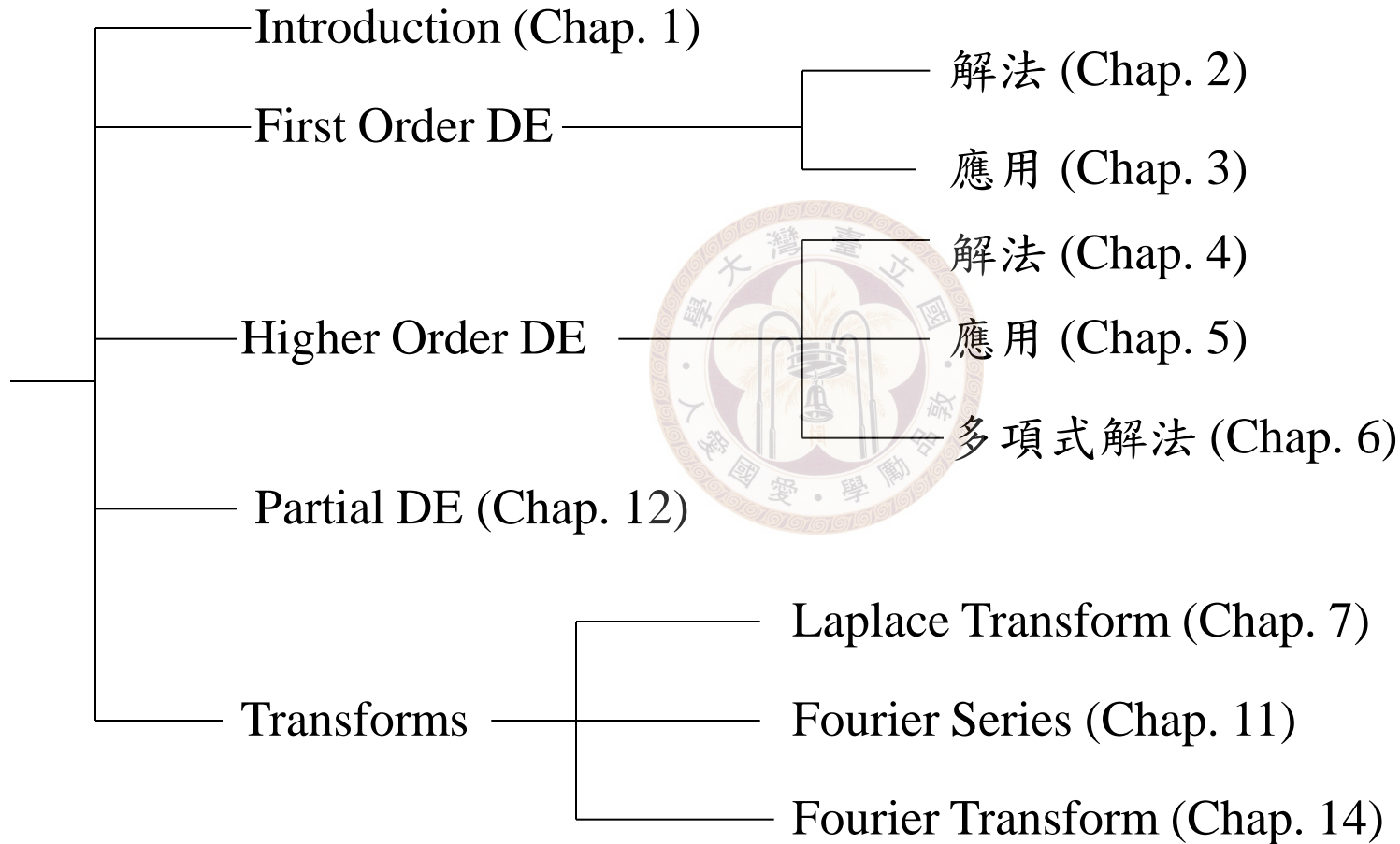
(4) 我週一至週四下午都在辦公室，有什麼問題，歡迎同學們來找我

# 上課日期

	Date (Wednesday, Friday)	Remark
1.	9/12, 9/14	
2.	9/19, 9/21	
3.	9/26, 9/28	
4.	10/3, 10/5	
5.	10/12	10/10 國慶
6.	10/17, 10/19	
7.	10/24, 10/26	
8.	10/31, 11/2	
9.	11/7: Midterm; (Chaps.1-5), 11/9	範圍： (Chaps.1-5)
10.	11/14, 11/16	
11.	11/21, 11/23	
12.	11/28, 11/30	
13.	12/5, 12/7	

14.	12/12, 12/14	
15.	12/19, 12/21	
16.	12/26, 12/28	
17.	1/2, 1/4	
18.	1/9 Finals	範圍：(Chaps. 6, 7, 11, 12, 14)





# Chapter 1 Introduction to Differential Equations

## 1.1 Definitions and Terminology (術語)

(1) **Differential Equation (DE)**: any equation containing derivation  
(page 2, definition 1.1)

$$\frac{dy(x)}{dx} = 1$$

$x$ : independent variable 自變數  
 $y(x)$ : dependent variable 應變數

$$\int_0^x \sin(2\pi t) f(t) dt + \frac{d^3 f(x)}{dx^3} = g(x)$$

- Note: In the text book,  $f(x)$  is often simplified as  $f$

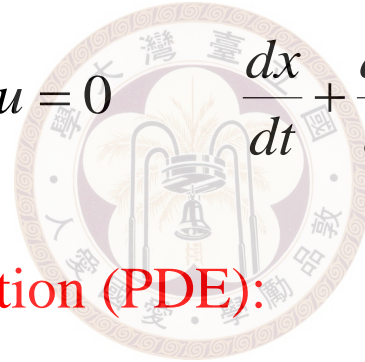
- notations of differentiation

$\frac{df}{dx}$	$\frac{d^2 f}{dx^2}$	$\frac{d^3 f}{dx^3}$	$\frac{d^4 f}{dx^4}$	, .....	Leibniz notation
$f'$	$f''$	$f'''$	$f^{(4)}$	, .....	prime notation
$\dot{f}$	$\ddot{f}$	$\overset{\cdot\cdot}{f}$	$\overset{\cdot\cdot\cdot}{f}$	, .....	dot notation
$f_x$	$f_{xx}$	$f_{xxx}$	$f_{xxxx}$	, .....	subscript notation

## (2) Ordinary Differential Equation (ODE):

differentiation with respect to **one independent variable**

$$\frac{d^3 u}{dx^3} + \frac{d^2 u}{dx^2} + \frac{d}{dx} u + \cos(6x)u = 0 \quad \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} = 2xy + z$$



## (3) Partial Differential Equation (PDE):

differentiation with respect to **two or more independent variables**

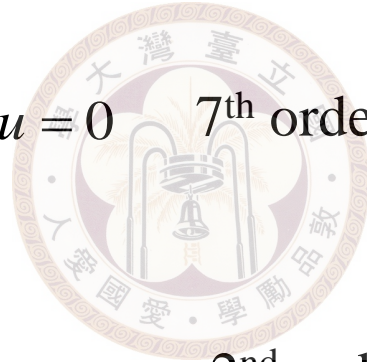
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \frac{\partial x}{\partial t} = \frac{\partial y}{\partial \tau}$$



**(4) Order of a Differentiation Equation:** the order of the highest derivative in the equation

$$\frac{d^7 u}{dx^7} + 2 \frac{d^6 u}{dx^6} + 3 \frac{du}{dx} + u = 0 \quad 7^{\text{th}} \text{ order}$$

$$\frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} - 4y = e^x \quad 2^{\text{nd}} \text{ order}$$



## (5) Linear Differentiation Equation:

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = g(x)$$

All the coefficient terms are independent of y.

Property of linear differentiation equations:

If  $a_n(x) \frac{d^n y_1}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y_1}{dx^{n-1}} + \cdots + a_1(x) \frac{dy_1}{dx} + a_0(x) y_1 = g_1(x)$

$$a_n(x) \frac{d^n y_2}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y_2}{dx^{n-1}} + \cdots + a_1(x) \frac{dy_2}{dx} + a_0(x) y_2 = g_2(x)$$

and  $y_3 = by_1 + cy_2$ , then

$$a_n(x) \frac{d^n y_3}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y_3}{dx^{n-1}} + \cdots + a_1(x) \frac{dy_3}{dx} + a_0(x) y_3 = bg_1(x) + cg_2(x)$$

## (6) Non-Linear Differentiation Equation

$$(y + 3) \frac{d^2 y}{dx^2} + \frac{dy}{dx} + 2y = x$$

$$\frac{d^2 y}{dx^2} + \frac{dy}{dx} + y^2 = e^x$$

$$\frac{d^2 y}{dx^2} + \frac{dy}{dx} + e^y = e^x$$



## (7) Explicit Solution (page 6)

The solution is expressed as  $y = \phi(x)$

## (8) Implicit Solution (page 7)

Example:  $\frac{dy^2}{dx} = -x$  ,



Solution:  $\frac{1}{2}x^2 + y^2 = c$  (implicit solution)

or  $y = \sqrt{c - x^2 / 2}$   
 $y = -\sqrt{c - x^2 / 2}$  (explicit solution)

## 1.2 Initial Value Problem (IVP)

A differentiation equation always has more than one solution.

for  $\frac{dy}{dx} = 1$  ,

$y = x$ ,  $y = x+1$  ,  $y = x+2$  ... are all the solutions of the above differentiation equation.

General form of the solution:  $y = x + c$ , where  $c$  is any constant.

The **initial value** (未必在  $x = 0$ ) is helpful for obtain the unique solution.

$$\frac{dy}{dx} = 1 \quad \text{and} \quad y(0) = 2 \quad \longrightarrow \quad y = x+2$$

$$\frac{dy}{dx} = 1 \quad \text{and} \quad y(2) = 3.5 \quad \longrightarrow \quad y = x+1.5$$

The  $k^{\text{th}}$  order differential equation usually requires  $k$  initial conditions (or  $k$  boundary conditions) to obtain the unique solution.

$$\frac{d^2 y}{dx^2} = 1 \quad \text{solution: } y = x^2/2 + bx + c,$$

$b$  and  $c$  can be any constant

$$y(1) = 2 \quad \text{and} \quad y(2) = 3 \quad (\text{boundary conditions, 在不同點})$$

$$y(0) = 1 \quad \text{and} \quad y'(0) = 5 \quad (\text{initial conditions})$$

$$y(0) = 1 \quad \text{and} \quad y'(3) = 2 \quad (\text{boundary conditions, 在不同點})$$

For the  $k^{\text{th}}$  order differential equation, the initial conditions can be  $0^{\text{th}} \sim (k-1)^{\text{th}}$  derivatives at some points.

# 1.3 Differential Equations as Mathematical Model

Physical meaning of **differentiation**:

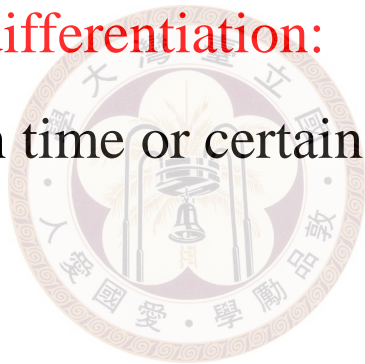
the variation at certain time or certain place

Example 1:

$$\frac{dA(t)}{dt} = kA(t)$$

A: population

人口增加量和人口呈正比



Example 2:

$$\frac{dT}{dt} = k(T - T_m)$$

$T$ : 熱開水溫度,

$T_m$ : 環境溫度

$t$ : 時間





大一微積分所學的：

$$\int f(t) dt \quad \text{的解}$$

$$\text{例如：} \int \frac{1}{t} dt = \ln|t| + c$$

$$\frac{dA(t)}{dt} = \frac{1}{t} \quad \longrightarrow \quad A(t) = \ln|t| + c$$

問題：

$$\int \frac{1}{t^2 + 4} dt = ?$$

- (1) 若等號兩邊都出現 dependent variable (如 pages 15, 16 的例子)
- (2) 若 order of DE 大於 1

$$\frac{d^2 A(t)}{dt^2} + 2 \frac{d A(t)}{dt} = 1$$

## Review

- dependent variable and independent variable
- DE
- PDE and ODE
- Order of DE
- linear DE and nonlinear DE
- explicit solution and implicit solution
- initial value
- IVP

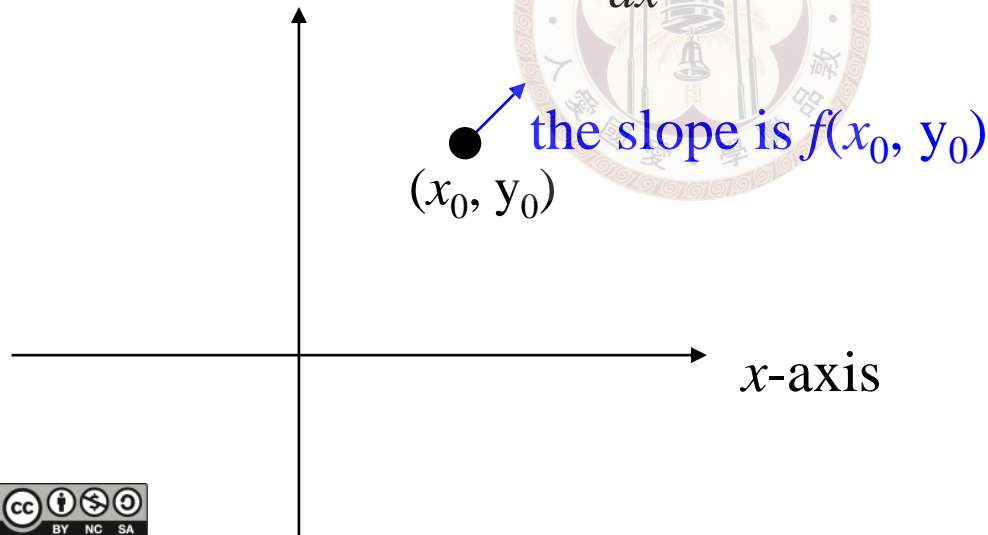


# Chapter 2 First Order Differential Equation <sup>19</sup>

## 2-1 Solution Curves without a Solution

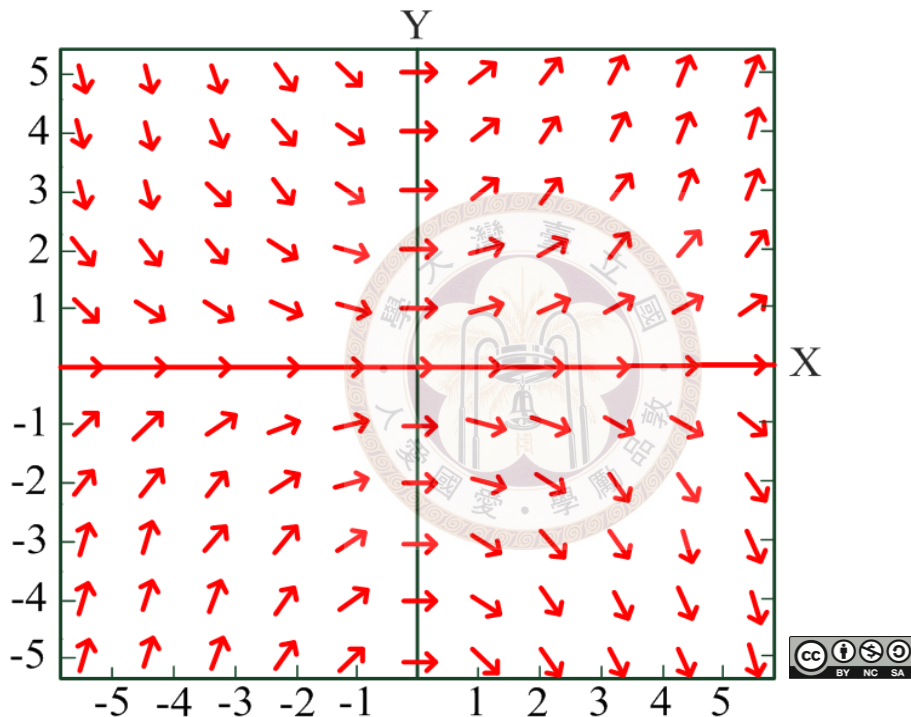
Instead of using analytic methods, the DE can be solved by graphs (圖解)

slopes and the field directions:  $\frac{dy}{dx} = f(x, y)$



Example 1

$$dy/dx = 0.2xy$$

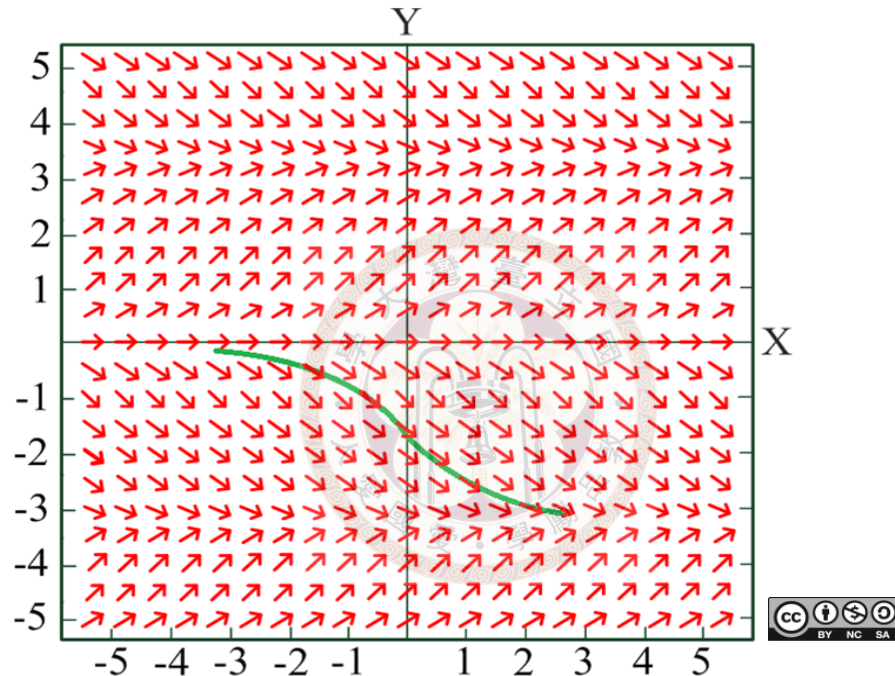


資料來源： Fig. 2-1-3(a) of “Differential Equations-with Boundary-Value Problem”, 7<sup>th</sup> ed., Dennis G. Zill and Michael R. Cullen.

## Example 2

$$dy/dx = \sin(y), \quad y(0) = -3/2$$

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資料來源： Fig. 2-1-4 of “Differential Equations-with Boundary-Value Problem”, 7<sup>th</sup> ed., Dennis G. Zill and Michael R. Cullen.

With initial conditions, one curve can be obtained

### Advantage:

It can solve some 1<sup>st</sup> order DEs that cannot be solved by mathematics.

### Disadvantage:

It can only be used for the case of the 1st order DE.

It requires a lot of time



# Section 2-6 A Numerical Method

- Another way to solve the DE without analytic methods

• independent variable  $x$   $\xrightarrow{\text{sampling(取樣)}}$   $x_0, x_1, x_2, \dots\dots\dots$

• Find the solution of  $\frac{dy(x)}{dx} = f(x, y)$

Since  $\frac{dy(x)}{dx} = f(x, y)$   $\xrightarrow{\text{approximation}}$   $\frac{y(x_{n+1}) - y(x_n)}{x_{n+1} - x_n} = f(x_n, y(x_n))$

$$y(x_{n+1}) = y(x_n) + f(x_n, y(x_n))(x_{n+1} - x_n)$$

前一點的值

取樣間格

- Example:
- $dy(x)/dx = 0.2xy \longrightarrow y(x_{n+1}) = y(x_n) + 0.2x_n y(x_n) * (x_{n+1} - x_n).$
- $dy/dx = \sin(x) \longrightarrow y(x_{n+1}) = y(x_n) + \sin(x_n) * (x_{n+1} - x_n).$

後頁為  $dy/dx = \sin(x)$ ,  $y(0) = -1$ ,

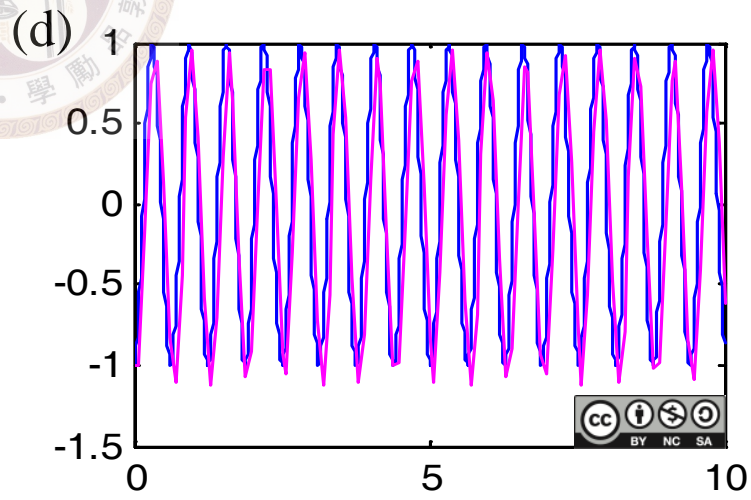
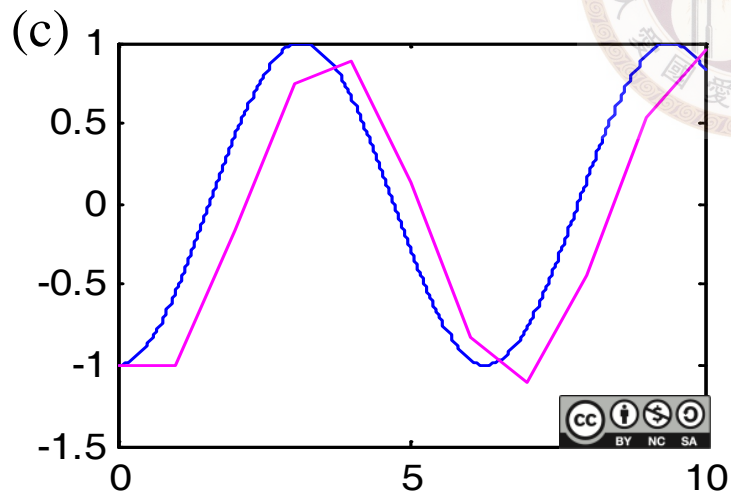
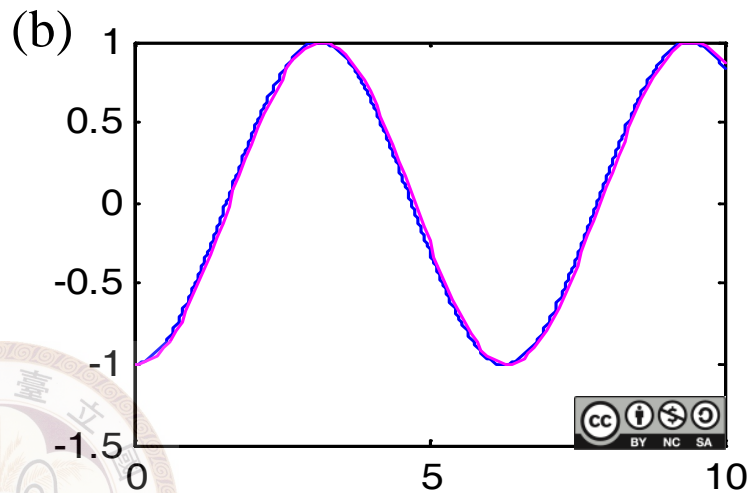
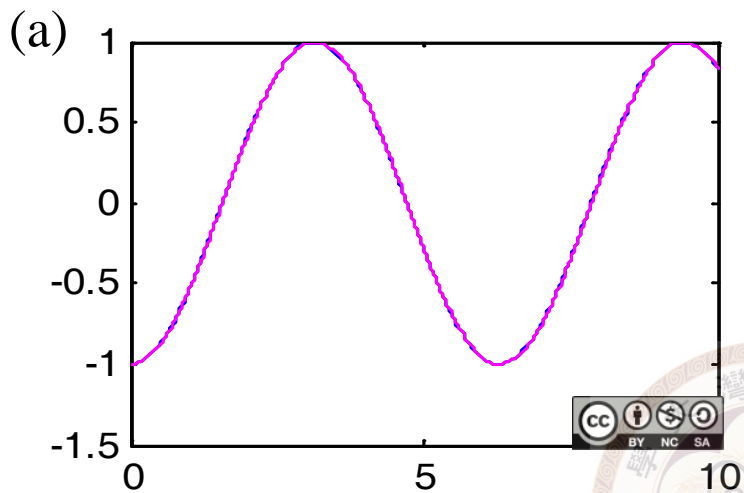
(a)  $x_{n+1} - x_n = 0.01$ ,      (b)  $x_{n+1} - x_n = 0.1$ ,

(c)  $x_{n+1} - x_n = 1$ ,      (d)  $x_{n+1} - x_n = 0.1$ ,  $dy/dx = 10\sin(10x)$  的例子

Constraint for obtaining accurate results:

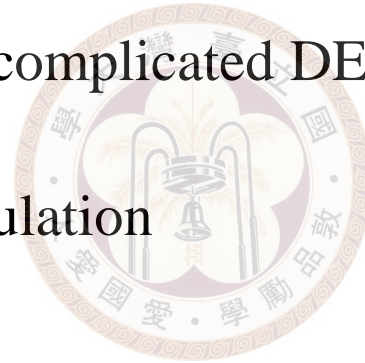
- (1) small sampling interval    (2) small variation of  $f(x, y)$





## Advantages

- It can solve some 1st order DEs that cannot be solved by mathematics.
- can be used for solving a complicated DE (not constrained for the 1<sup>st</sup> order case)
- suitable for computer simulation



## Disadvantages

- numerical error (數值方法的課程對此有詳細探討)

## Exercises for Practicing

(not homework, but are encouraged to practice)

1-1: 1, 13, 19, 23, 33

1-2: 3, 13, 21, 33

1-3: 2, 7, 28

2-1: 1, 13, 20, 25, 33

2-6: 1, 3



頁碼	作品	版權圖示	來源/作者
19			<p>台灣大學 電信工程研究所 丁建均教授  以創用CC「姓名標示-非商業性-相同方式分享」臺灣3.0版授權釋出。</p>
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