Introduction to General Relativity

Einstein's 1915 relativistic theory of gravity, general relativity, will soon be a century old. It is the classical theory of one of the four fundamental forces. It underlies our contemporary understanding of the **big bang, black holes, pulsars, quasars, x-ray sources, the final destiny of stars, gravitational waves, and the evolution of the universe**. It is the intellectual origin of many of the current ideas in the quest for a unified theory of the fundamental forces. General relativity is one of the most beautiful and revolutionary ideas in modern science—the idea that **gravity *is* the geometry of curved four-dimensional spacetime**. General relativity and quantum mechanics are usually regarded as the two greatest developments of 20th century physics.

J. B. Hartle, Am. J. Phys. 74, 14 (2006)

Key ideas in general relativity

(1) ***Gravity is Geometry***. Phenomena familiarly seen as arising from gravitational forces in a Newtonian context are more generally due to the curvature of the geometry of four-dimensional spacetime.

(2) ***Mass-Energy is the Source of the Spacetime Curvature***. Mass is the source of the spacetime curvature and, because general relativity incorporates special relativity, any form of energy is also a source of the spacetime curvature.

(3) ***Free Mass Moves on Straight Paths in Curved Spacetime***. In general relativity the Earth moves around the Sun in its orbit not because of a gravitational force exerted by the Sun, but because it is following a straight path in the curved spacetime produced by the Sun.

J. B. Hartle, Am. J. Phys. 74, 14 (2006)

J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity*

S. Carroll, *Spacetime and Geometry: An introduction to General Relativity*

(http://arxiv.org/PS_cache/gr-qc/pdf/9712/9712019v1.pdf)

G. 't Hooft, *Introduction to General Relativity*

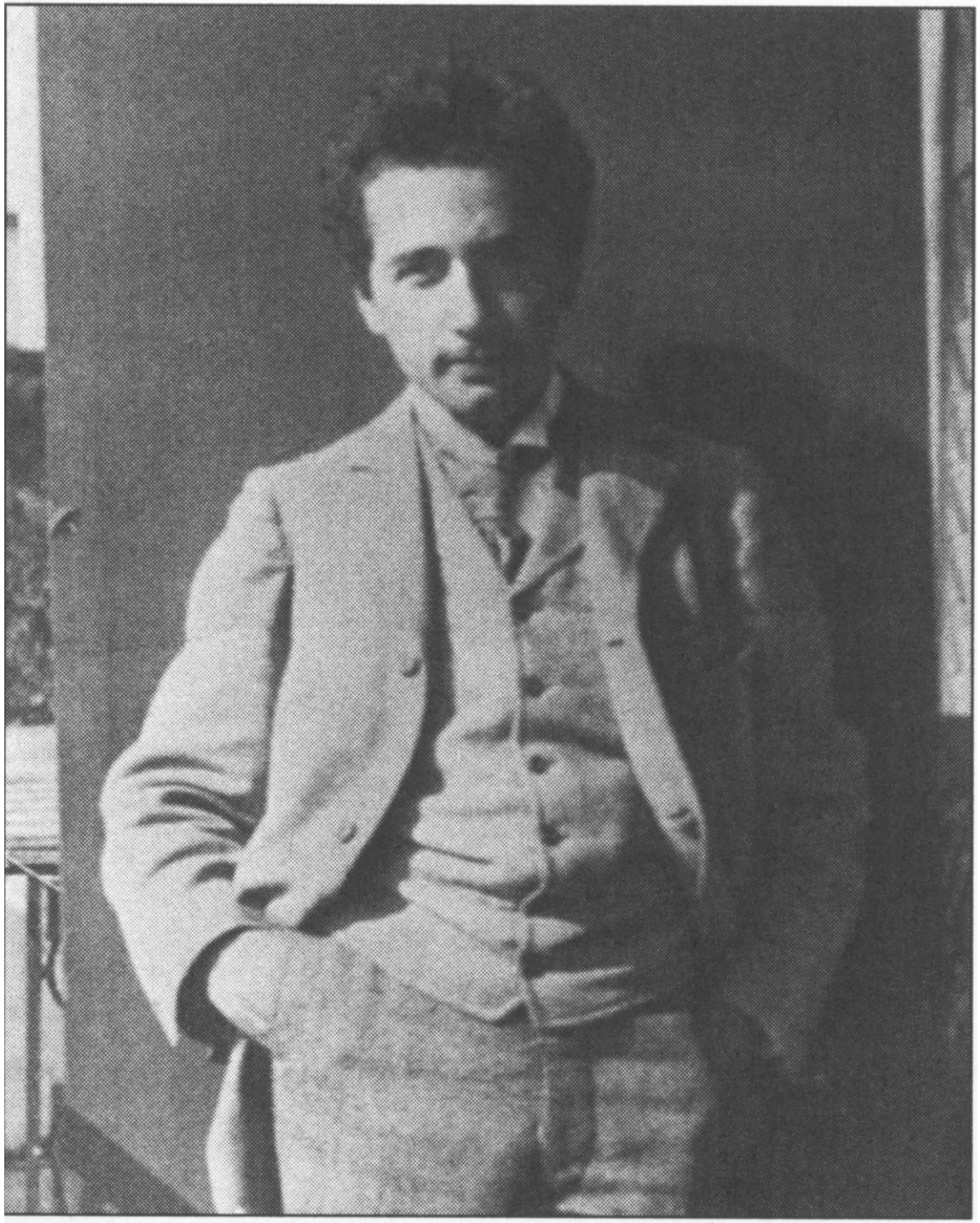
(http://www.phys.uu.nl/~thooft/lectures/genrel.pdf)



1898

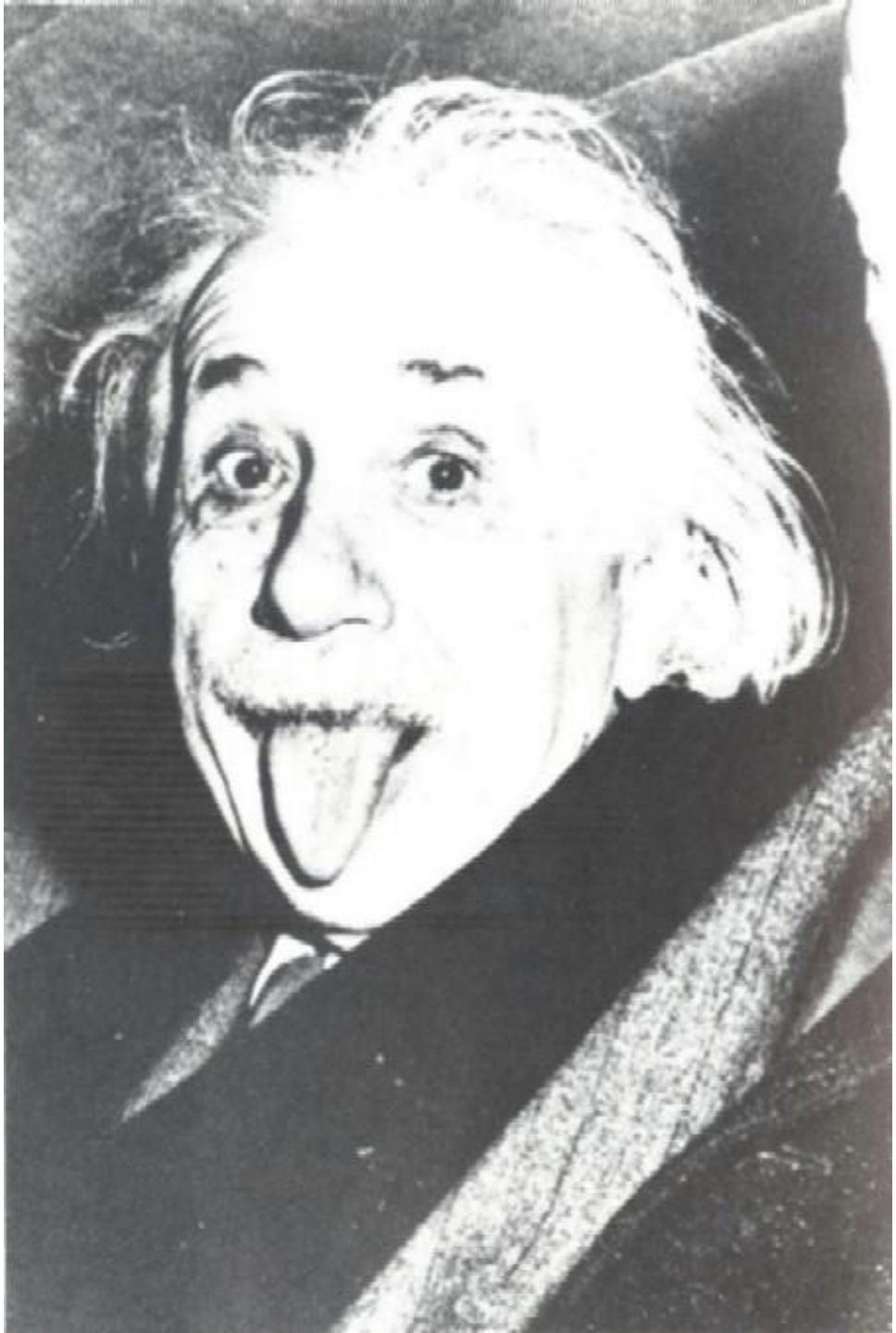
19歲

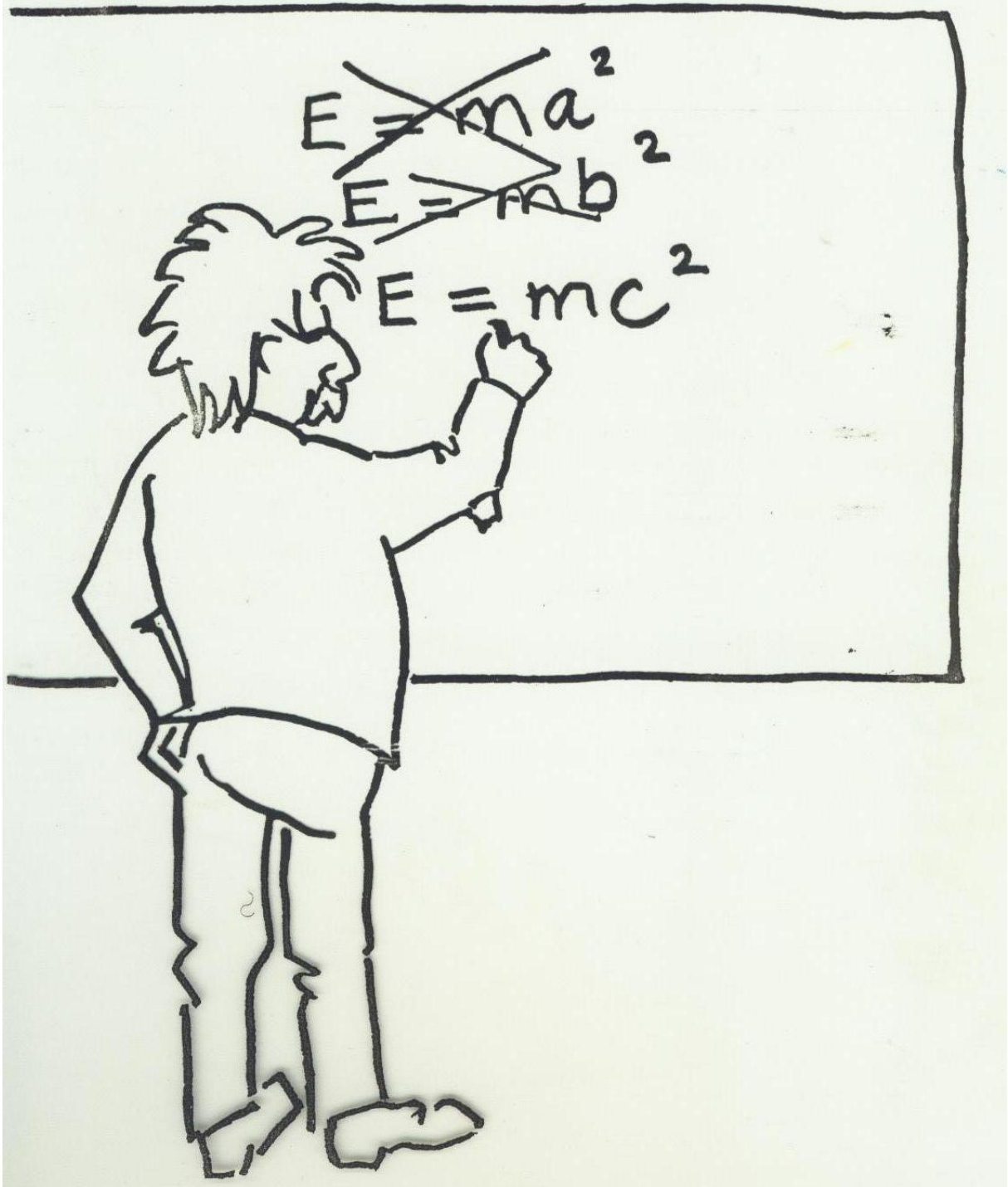
蘇黎士
技術學院





M. Grossmann




E = ma²
E = mb²
E = mc²

愛因斯坦提出**狹義相對論**的動機在於解決一個困惑他近十年的矛盾：

這個矛盾就是**相對性原理**與**馬克斯威爾電磁理論**在表面上似乎是不相容的。



在高速飛行的客機中

(或是在船上，或是在火車上・・・)



就好像在地面上一樣？Yes! 這是大家熟悉的**事實！**

完全一樣？Yes!

為什麼這樣？**沒人知道!**

這件「顯而易見」的事實其實是大自然很奧妙的本性之一。

物理上我們稱無法再往下深究為什麼的**事實**為**原理**。

相對性原理

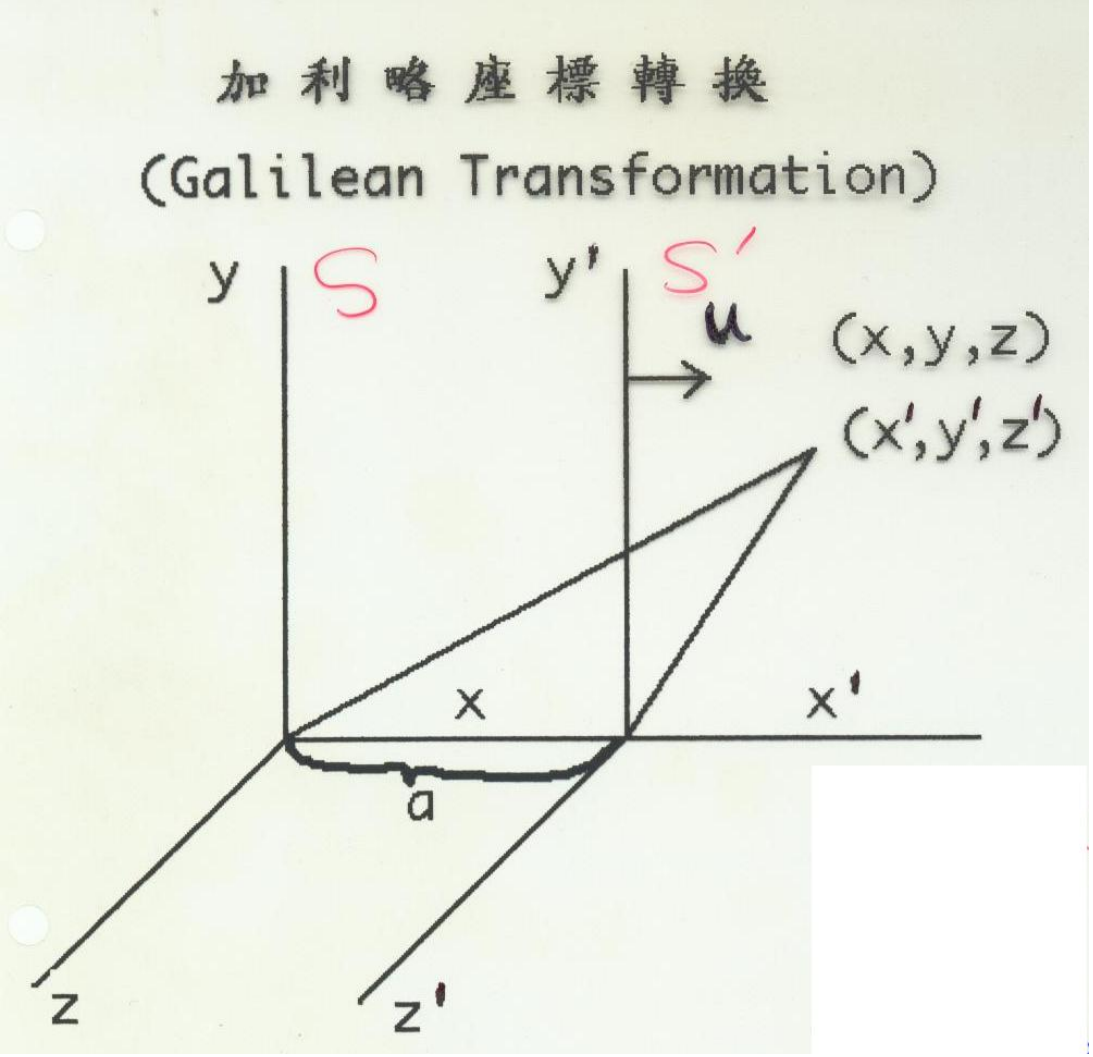
Principle of Relativity

Galileo Galilei (1564-1642) **加利略**

慣性

物理定律對於每一個慣性座標系中的觀察者而言—
無論她是靜止的或是以固定速度前進—
都是一樣的。
→對稱

所有的觀察者都是「**等價**」的
沒有「絕對」運動



$a = u\,t$

$$x' = x - u\,t$$
$$y' = y$$
$$z' = z$$

$$t' = t$$

空間座標會依觀測者而異

但是時間卻是**絕對**的(時間不是座標)

$$v = \frac{\Delta x}{\Delta t}$$

$$v' = \frac{\Delta x'}{\Delta t'}$$

$$= \frac{\Delta x - u\,\Delta t}{\Delta t}$$

$$= \frac{\Delta x}{\Delta t} - u$$

$\Delta x' = \Delta x - \Delta(ut)$

$\Delta t' = \Delta t$

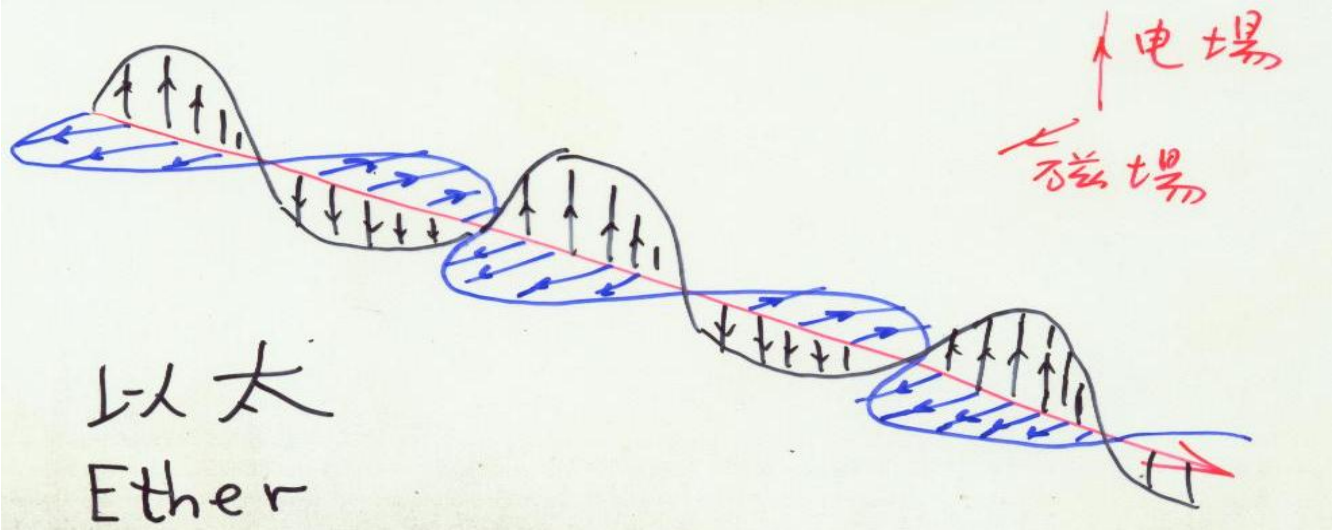
$$\boxed{v' = v - u}$$

$$\boxed{v = v' + u}$$

一切的速度都是相對的

不同的觀測者會測量到不同的速度

馬克斯威爾(J. C. Maxwell, 1831-1879)
在1860年代以四個數學方程式來描述一切電與磁的現象。
他並且將光解釋為電磁波，也就是電場與磁場的振盪。



馬克斯威爾還算出光速 (在以太座標)大約為每秒30萬公里

光速＝ c

馬克斯威爾認為光速只有在以太靜止座標才會等於c。如果我們相對於以太的速度不等於零，則我們所量到的光速就**不會**為c。

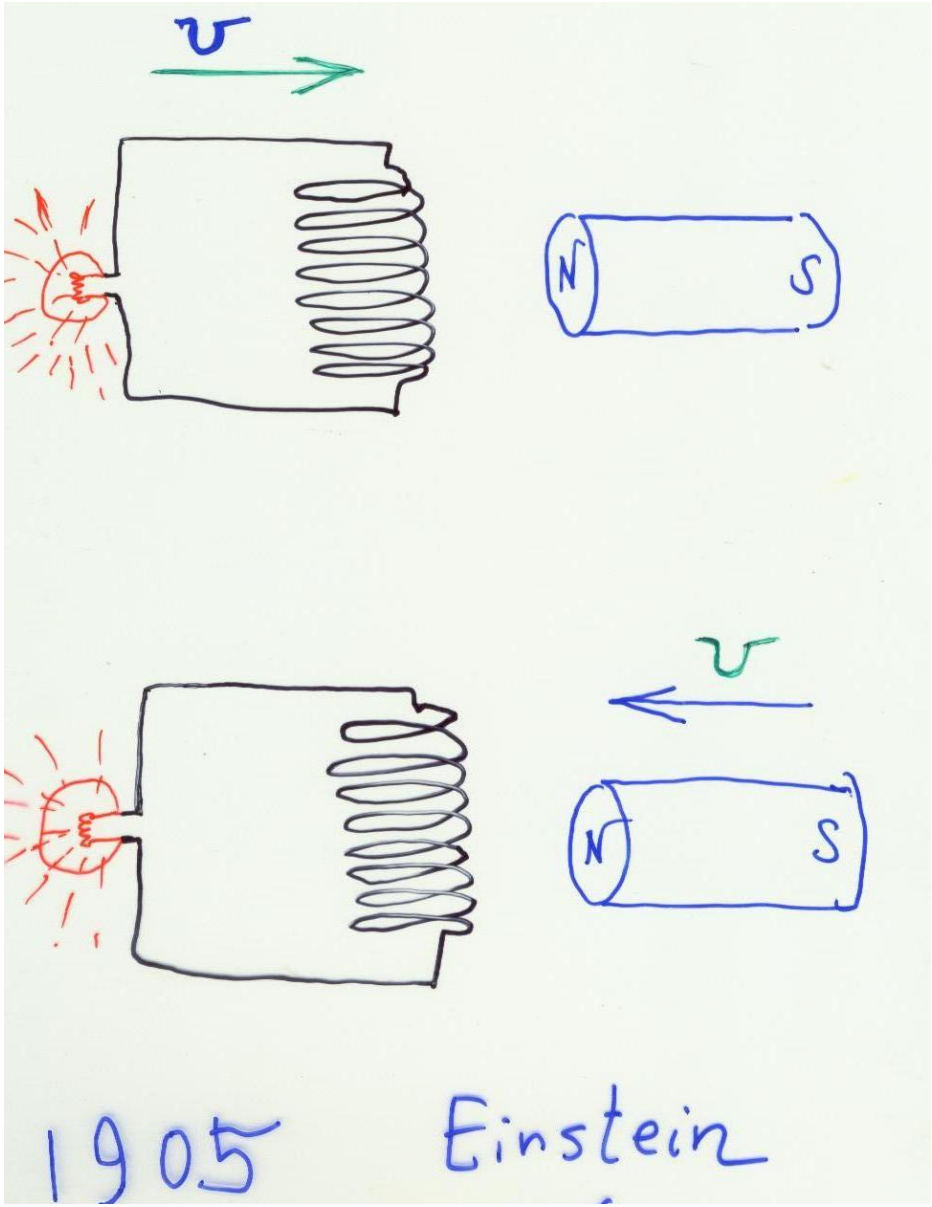
這是很「**合理**」的推斷。

但是實驗學家卻發現無論我們在什麼座標測量光速，我們所量到的光速**都等於c**。

這怎麼可能呢？

愛因斯坦有自己的理由認為光速在任何慣性座標中都等於c。

他的理由是：



既然馬克斯威爾方程式在每個(慣性)座標都**成立**，那麼每個座標系中的物理學家所計算而得的電磁波速度—也就是光速—就應該都等於c。

因此愛因斯坦假設：

狹義相對論(Special relativity)基本假設：

(1) 所有慣性座標系中的物理定律都有相同的形式。

— 沒有任何慣性座標系有特殊的地位

(相對性原理)

(2) 任何慣性座標系所測得的光速都是c。

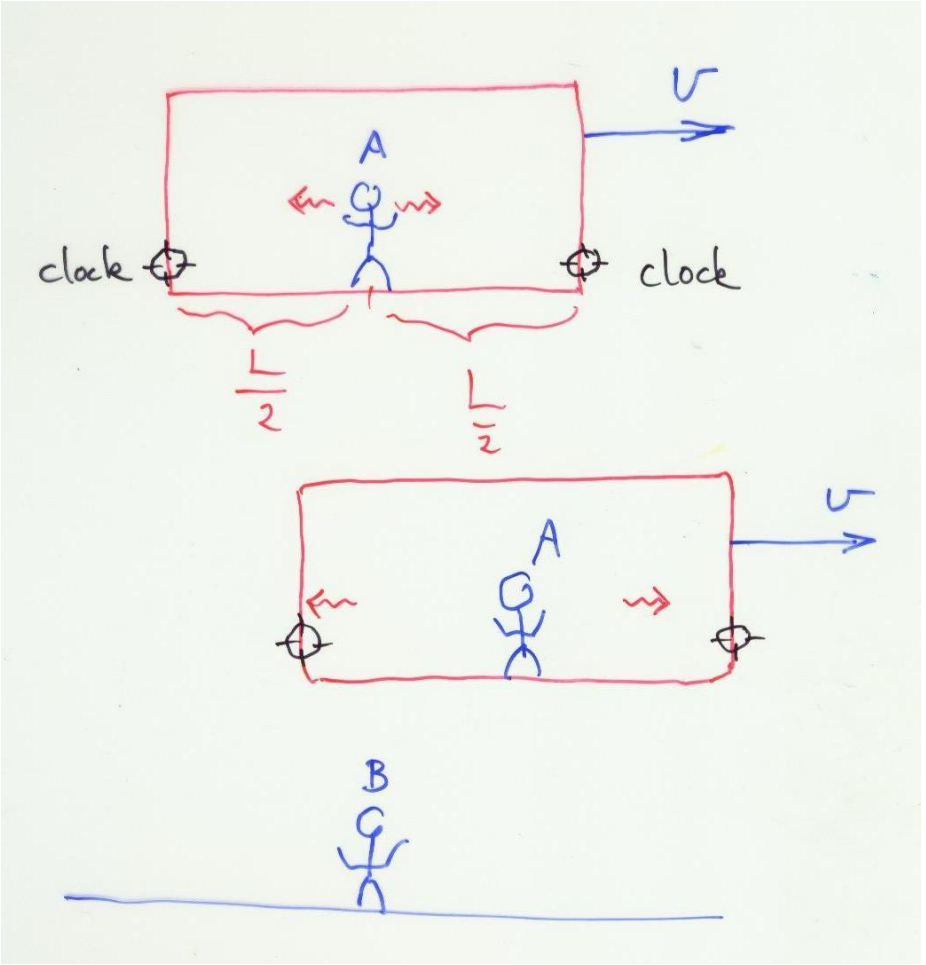
但是(2)與大家非常熟悉的加利略變換**相衝突**！

問題出在那裡？

愛因斯坦在高中時期(16歲！)就開始思考這一個的問題。他曾經想過如果能以光速前進，究竟會看到什麼？

他在1905年才領悟到問題出在「**同時**」(**時間**)的概念必須重新檢討。

對於觀測者A來說，位於車廂兩端的鐘必是同時接收到光。但是對於觀測者B來說，一個鐘會比另一個鐘更早接收到光。



「同時 (simultaneity)」是相對的概念！

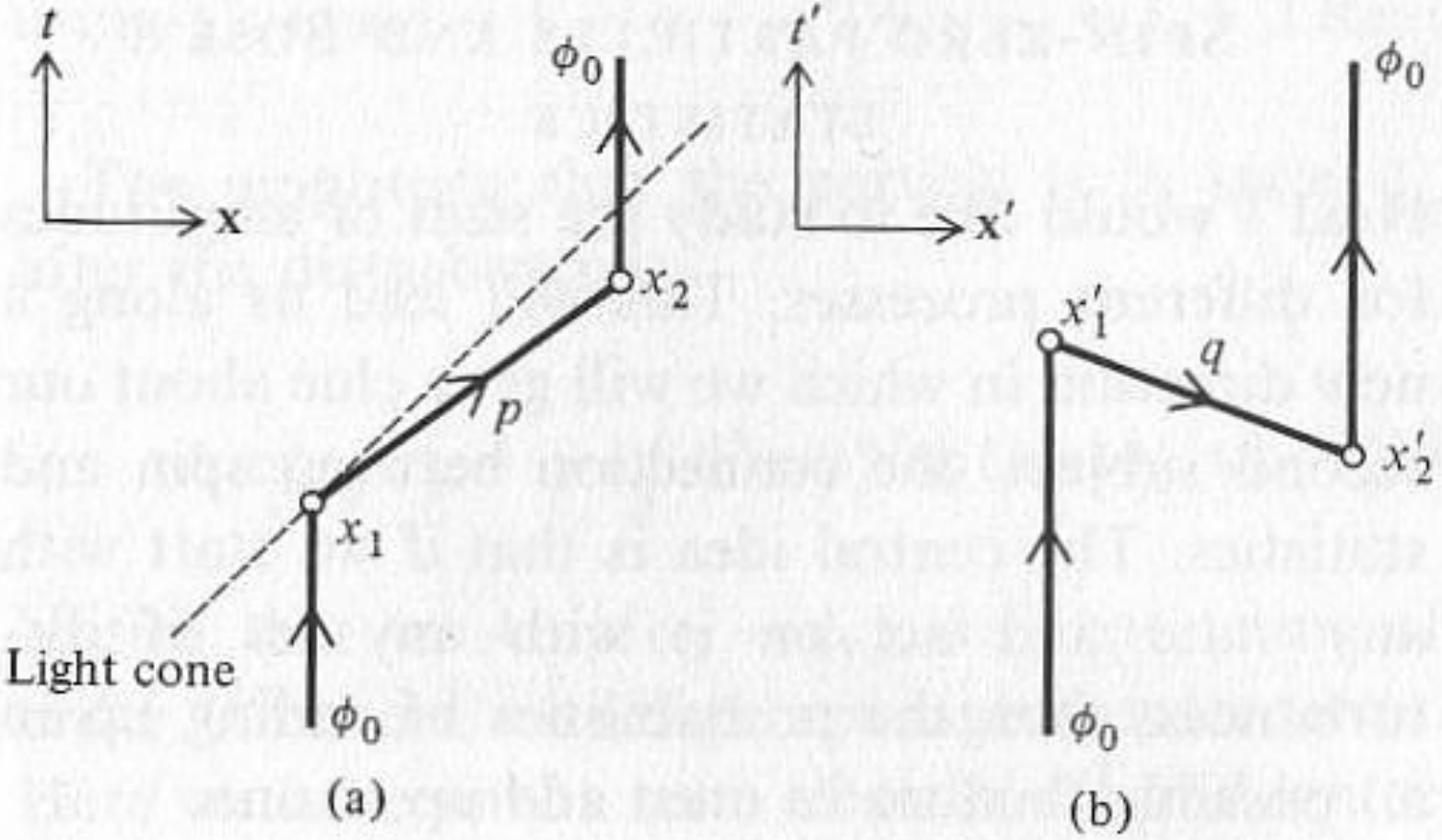
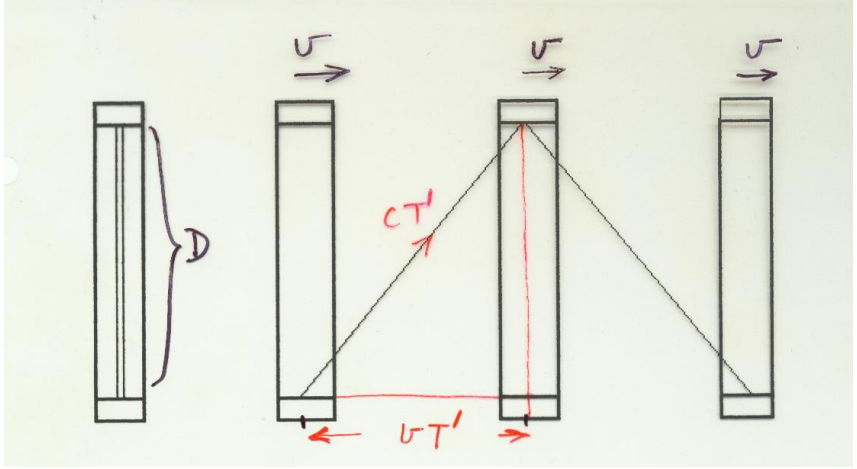


Fig. 2 The same process viewed from two different frames. (a) Original frame ($t_2 > t_1$); (b) moving frame ($t'_2 > t'_1$).



Sketch of P. A. M. Dirac by R. P. Feynman

光鐘



$$T = \frac{D}{c} \quad , \quad T' = ?$$

$$(cT')^2 = (vT')^2 + D^2 \quad , \quad (c^2 - v^2)T'^2 = D^2$$

畢氏定理

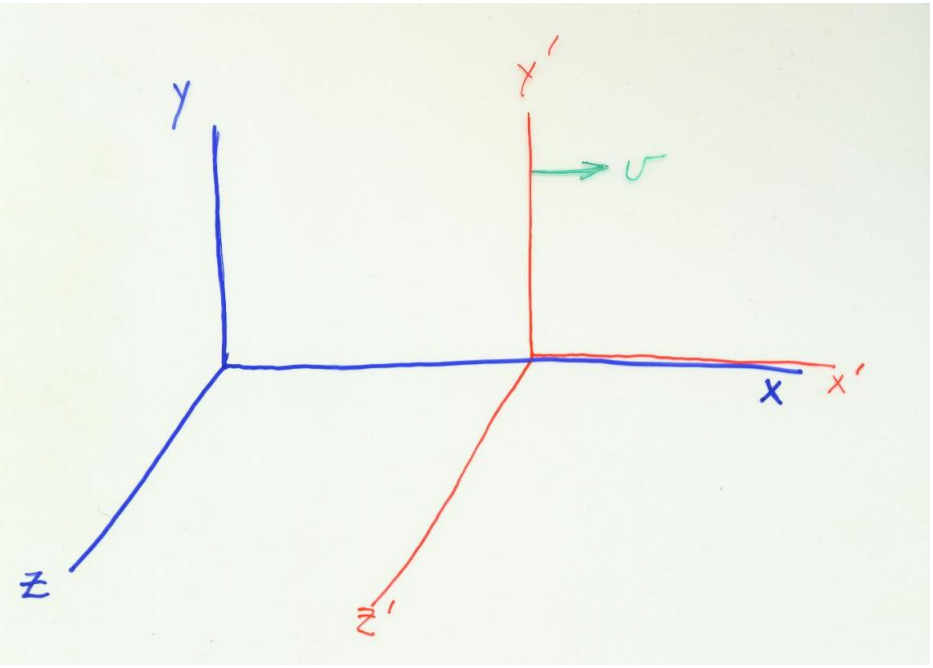
$$T' = \frac{D}{\sqrt{c^2 - v^2}} = \frac{D}{c}\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\boxed{T' = T\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} > T}$$

動的鐘走的比較慢！

時間膨脹

time dilation



$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

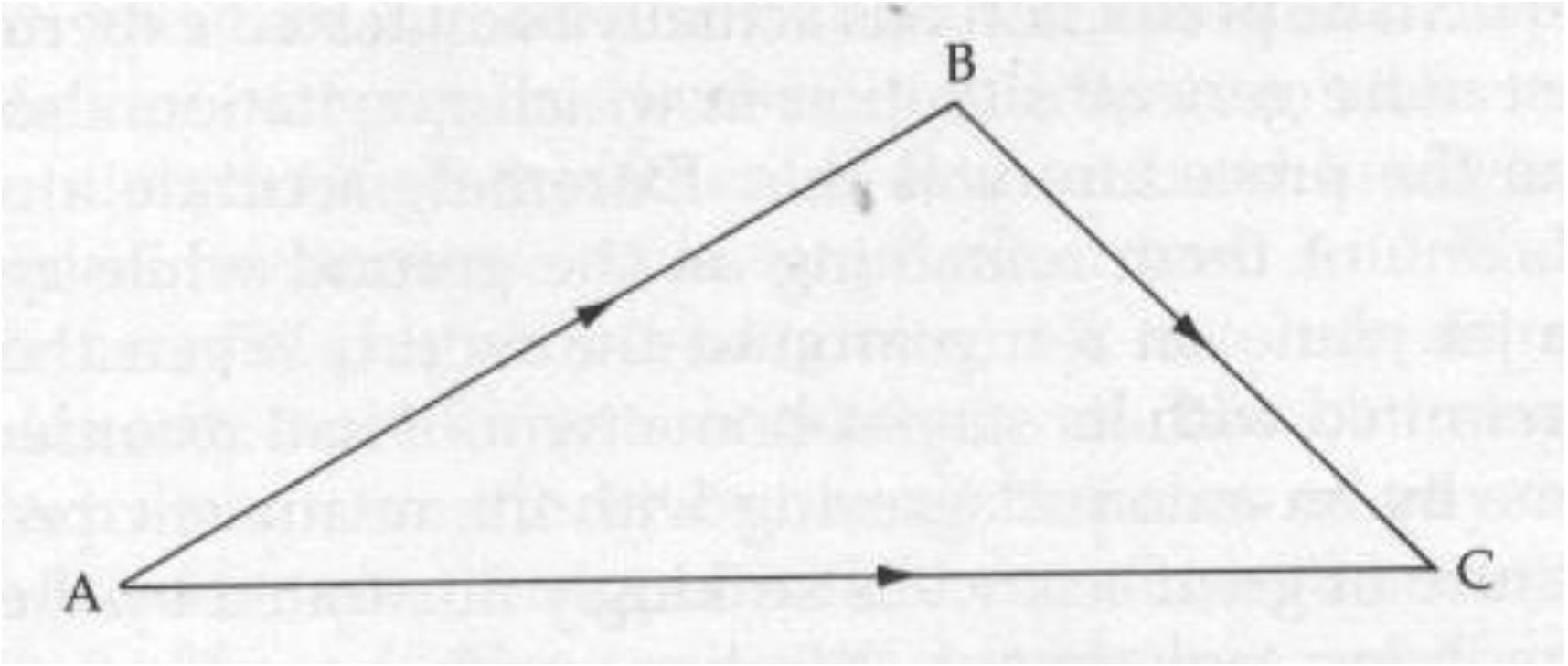
$$\gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

勞侖茲變換

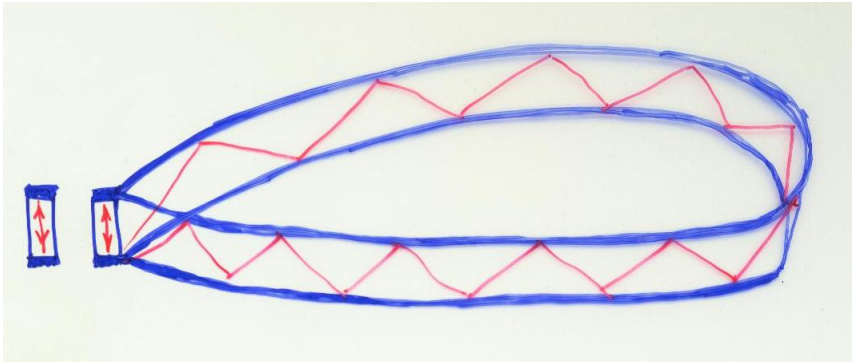
Lorentz Transformation

Lorentz Symmetry

Relativistic Invariance



The “Route-Dependence” of Time

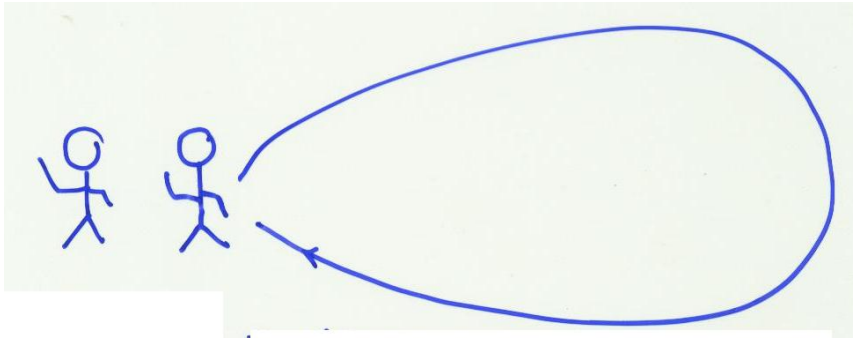


運動的光鐘跳動的次數較少

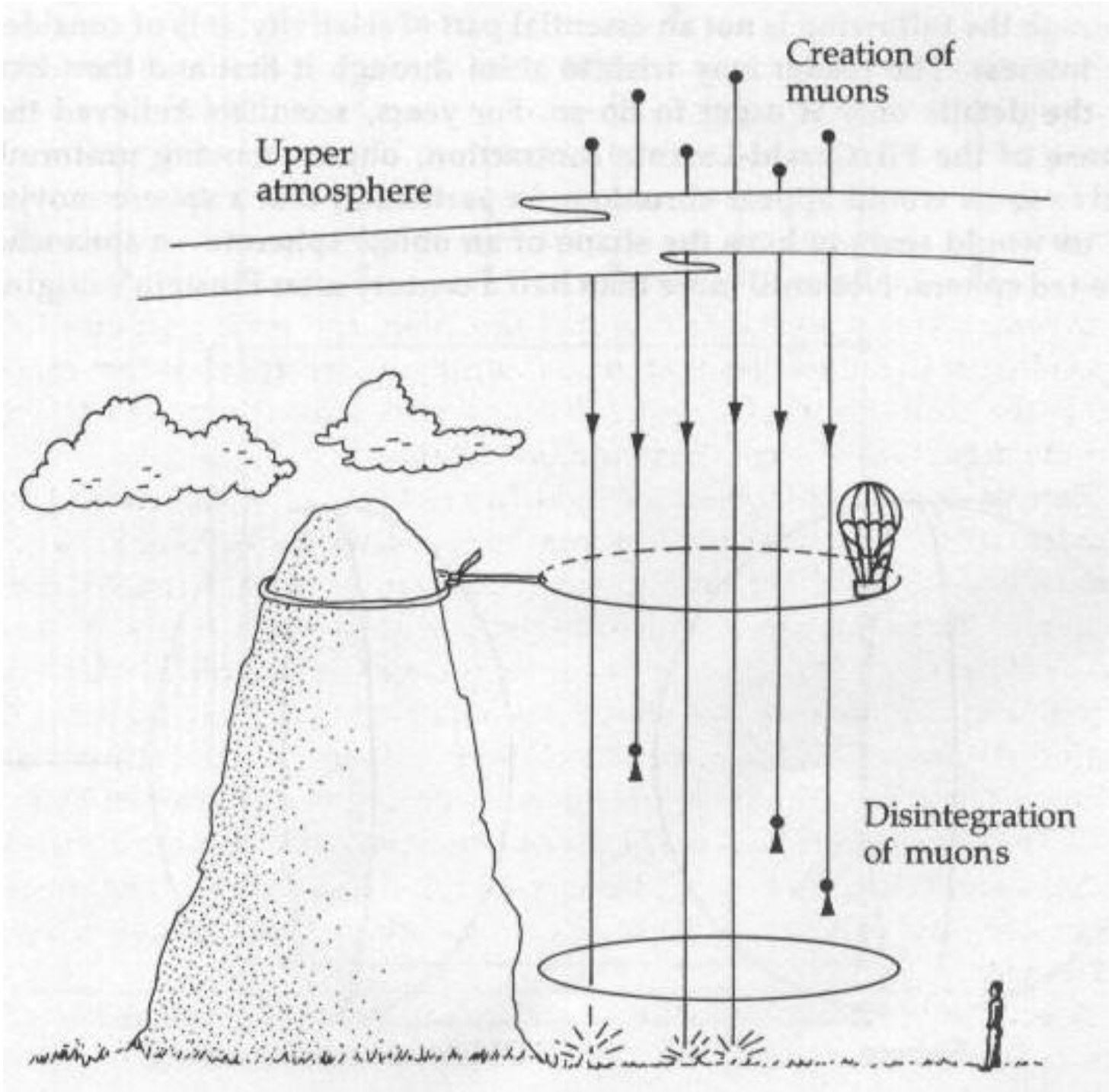
…人…心臟跳動的次數較少

實驗學家已經證實運動中的緲子壽命的確比較長

雙生姊妹



星際旅行一趟後回到地球上的人會比留在地球上的人年輕


Creation of
muons
Upper
atmosphere
Disintegration
of muons

時間成了座標之一，會隨觀測者而變。

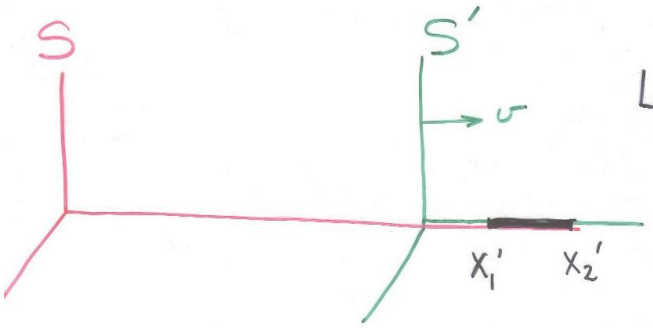
時間與空間不再是相互獨立的概念，我們必需將兩者結合起來，將它們看成是不可分割的一體

也就是說

空間	+	時間	→	時空
space	+	time	→	spacetime
3維		1維		4維

愛因斯坦的數學老師閔考斯基(Minkowski)首先強調4維時空的幾何意義，所以我們也常將4維時空稱為**閔考斯基時空**。

時間膨脹
time dilation

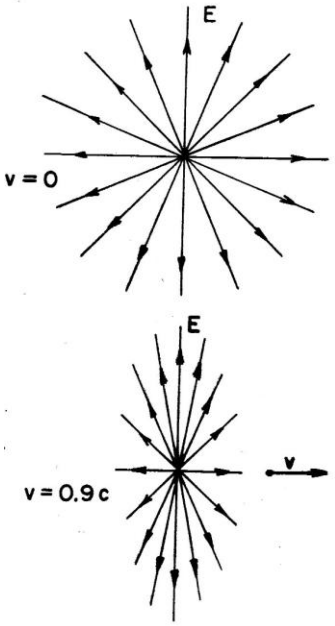


$$X_1' = \frac{X_1 - vt_1}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad X_2' = \frac{X_2 - vt_2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Length in S at time $t = t_1 = t_2$

$$L = X_2 - X_1$$

But $$X_2' - X_1' = \frac{X_2 - X_1}{\sqrt{1 - \frac{v^2}{c^2}}}$$



長度收縮
length contraction

$$L' = \frac{L}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\boxed{L = \sqrt{1 - \frac{v^2}{c^2}}\, L' < L'}$$

t, $\vec{x} = (x, y, z)$ $\longrightarrow$ $x^{\mu} = (t, x, y, z)$

3維向量 vector

4維向量

動量 $\vec{p} = (p_x, p_y, p_z)$ $\longrightarrow$ $p^{\mu} = (E, p_x, p_y, p_z)$

↑ 能量 energy

$p^\mu = (E, \vec{p})$ $\qquad (E', \vec{p}')$

$$E' = \frac{E - vp_x}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$p_x' = \frac{p_x - \frac{v}{c^2}E}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$p_y' = p_y$$

$$p_z' = p_z$$

$\vec{p}' = 0 \rightarrow p_x = \frac{v}{c^2}E$

$\downarrow$

$E' = \sqrt{1 - \frac{v^2}{c^2}}\,E$

$\downarrow$

$E = \frac{E'}{\sqrt{1 - \frac{v^2}{c^2}}}$

$\downarrow$ if $E' = mc^2$

$E = mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + \cdots$

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$E = m c^2$



撰文／高涌泉

理查・羅逖與「不影響考生作答」

如果有人在台灣宣揚羅逖提倡的實用主義，大約不會遇上任何阻力，
因為我們實在很熟悉「信念是依狀況而定的」這回事。

名哲學家理查・羅逖（Richard Rorty）於6月初過世了，享年75歲。《紐約時報》的訃聞說羅逖「在哲學、政治、文學理論等領域上的創見，使他成為世界上最具影響的當代思想家之一」，其他哲學家也用「最能以新觀點、新眼光、新理論來挑戰其同僚」、「勇敢、挑釁、令人興奮、富想像力、但又讓人極傷腦筋」之類的讚詞來追悼他。

我見過這位名人一回，那是在十多年前。那時羅逖訪問台灣，到台大哲學系演講，我恰好路過講廳，好奇地進去旁聽了一會。我一入門，即看到有趣的景象：白髮的羅逖穿西裝戴眼鏡坐在講廳前方的桌子後頭，手拿講稿，一字一句面向聽眾細聲誦讀；聽眾大半是看起來還有些青澀的學生，他們坐在講廳中間一排排的桌子旁，人人低頭用心做筆記，沒能正眼瞧羅逖。嘿，這樣好像不是領略哲學大師風範最好的方式吧！

我其實是在後來所謂的「科學戰爭」這場學術界風暴中，才開始認識羅逖「讓人極傷腦筋」的立場與文采。羅逖的哲學對手鄧奈特（Daniel Dennett）在悼文中講述了兩人的一段對話，非常傳神地點出了羅逖為何會在哲學家中獨樹一幟：「我（鄧奈特）說我非常在意科學家是否尊敬我，我認為將哲學議題以科學家能夠理解與欣賞的方式解釋給他們聽，是很重要的事。羅逖說他一點也不在乎科學家如何看待他的工作，他渴求的是詩人的注目與敬意。」

就哲學立場而論，羅逖所信仰的是實用主義、所仰慕的英雄是實用主義哲學家杜威（John Dewey）。實用主義者不承認有放諸四海皆準的原則可以幫我們判定何謂「真理」、何謂「善」，因此他們不會浪費時間去關心真理、善、或神的本質究竟為何的問題。對於羅逖來說，像「$E=mc^2$」、「2＋2＝4」之類的敘述都是真的，也是有用的，但是我們不必去追究這類陳述是不是對應到某個「實在」的某種固有性質。簡單的講，羅逖厭惡形上學。

自柏拉圖以來，西方哲學家所關注的問題就是真實與表象之間、發明與發現之間、相對與絕對之間、科學與非科學之間有何分野；羅逖受杜威的影響，不相信這些探索有什麼意義。如此的立場當然令主流哲學家「極傷腦筋」，也讓羅逖捲入了科學戰爭，與物理學家溫伯格（Steven Weinberg）辯論起來。羅逖的確認為我們的信念是「依狀況而定的」，沒有必然的對或錯可言，不過他拒絕因此被戴上「相對主義者」的帽子。

羅逖在美國提倡實用主義，之所以會引起爭議，原因在於普遍真理與理性等概念在西方社會有深厚的傳統。可是如果有人在台灣宣揚羅逖的理念，大約不會遇上任何阻力，因為我們實在很熟悉「信念是依狀況而定的」這回事。我最近看到一件精彩的例子，清楚地呈現了實用主義在台灣的主宰地位，值得在此一提：

今年第一次國中基測自然科第28題是：「蓉蓉發明一艘可以光速行進的太空船。已知有四顆星球與地球的距離分別為：超人星25光年，孫悟星1.7光年，寶貝星2.2天文單位，小咪星1.3天文單位。若蓉蓉欲搭光速太空船從地球出發前往上述星球，則1小時之內可到達的星球有哪些？（光年：光走一年的距離；天文單位：地球到太陽的距離，約為光走500秒的距離）」

國中基測推動工作委員會所公佈的答案是寶貝星與小咪星（大家稍微想一下就可了解為何如此），可惜這個答案是錯的！因為如果蓉蓉真的能夠以光速前進，依據狹義相對論，四顆星球皆可以在一瞬間（以蓉蓉太空船內的鐘為準）抵達。顯然出題者忽略了愛因斯坦的理論。有人將此事告知該單位，沒想到他們回答說，本題雖有「不周延之處，但不影響考生作答」，所以無須更正答案，亦即他們假設國中生不懂相對論！換句話說，本題的答案得看答題者是誰而定，完全符合實用主義的精神。 SA

高涌泉　台灣大學物理系教授

今年第一次國中基測自然科第28題是：「蓉蓉發明一艘可以光速行進的太空船。已知有四顆星球與地球的距離分別為：超人星25光年，凱蒂星1.7光年，寶貝星2.2天文單位，小咪星1.3天文單位。若蓉蓉欲搭光速太空船從地球出發前往上述星球，則1小時之內可到達的星球有哪些？（光年：光走一年的距離；天文單位：地球到太陽的距離，約為光走500秒的距離）」

國中基測推動工作委員會所公佈的答案是寶貝星與小咪星（大家稍微想一下就可了解為何如此），可惜這個答案是錯的！因為如果蓉蓉真的能夠以光速前進，依據狹義相對論，四顆星球皆可以在一瞬間（以蓉蓉太空船內的鐘為準）抵達。顯然出題者忽略了愛因斯坦的理論。有人將此事告知該單位，沒想到他們回答說，本題雖有「不周延之處，但不影響考生作答」，所以無須更正答案，亦即他們假設國中生不懂相對論！換句話說，本題的答案得看答題者是誰而定，完全符合實用主義的精神。

不過狹義相對論還不能全然令人滿意，

因為我們只考慮了慣性座標系

以及勞侖茲變換

(座標之間的相對速度必需是固定的)。

如果座標之間有相對加速度，

則會如何呢？

我們是不是可以進一步要求

物理定律的形式在任意的座標轉換下

都是不變的？

此外我們還沒有將重力考慮進來。

在狹義相對論中，光速是訊息(交互作用)傳遞速度的上限。

但是牛頓的萬有引力(重力)是**瞬時力**，

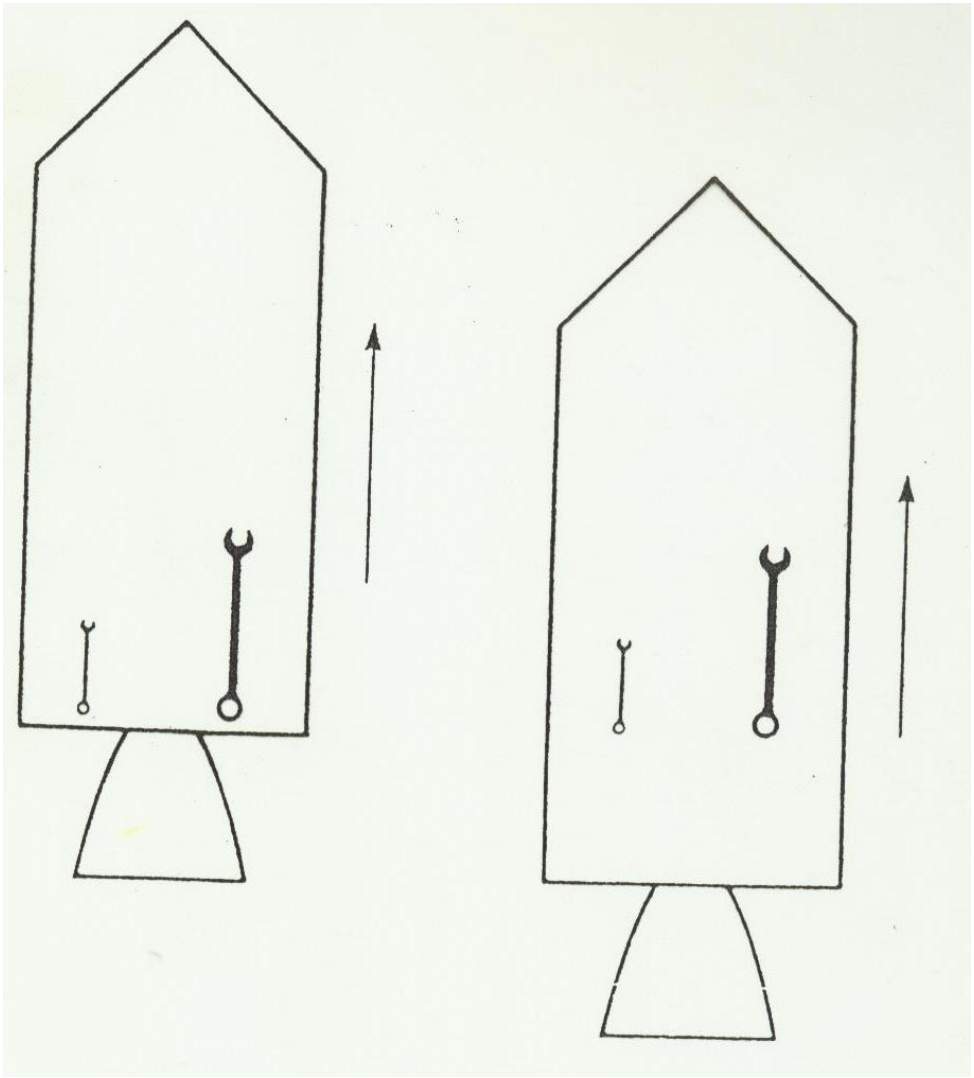
所以牛頓的重力理論與狹義相對論**不相容**。

我們該如何建造一個與狹義相對論原則相容的重力論？

”我這一輩子最快樂的點子“

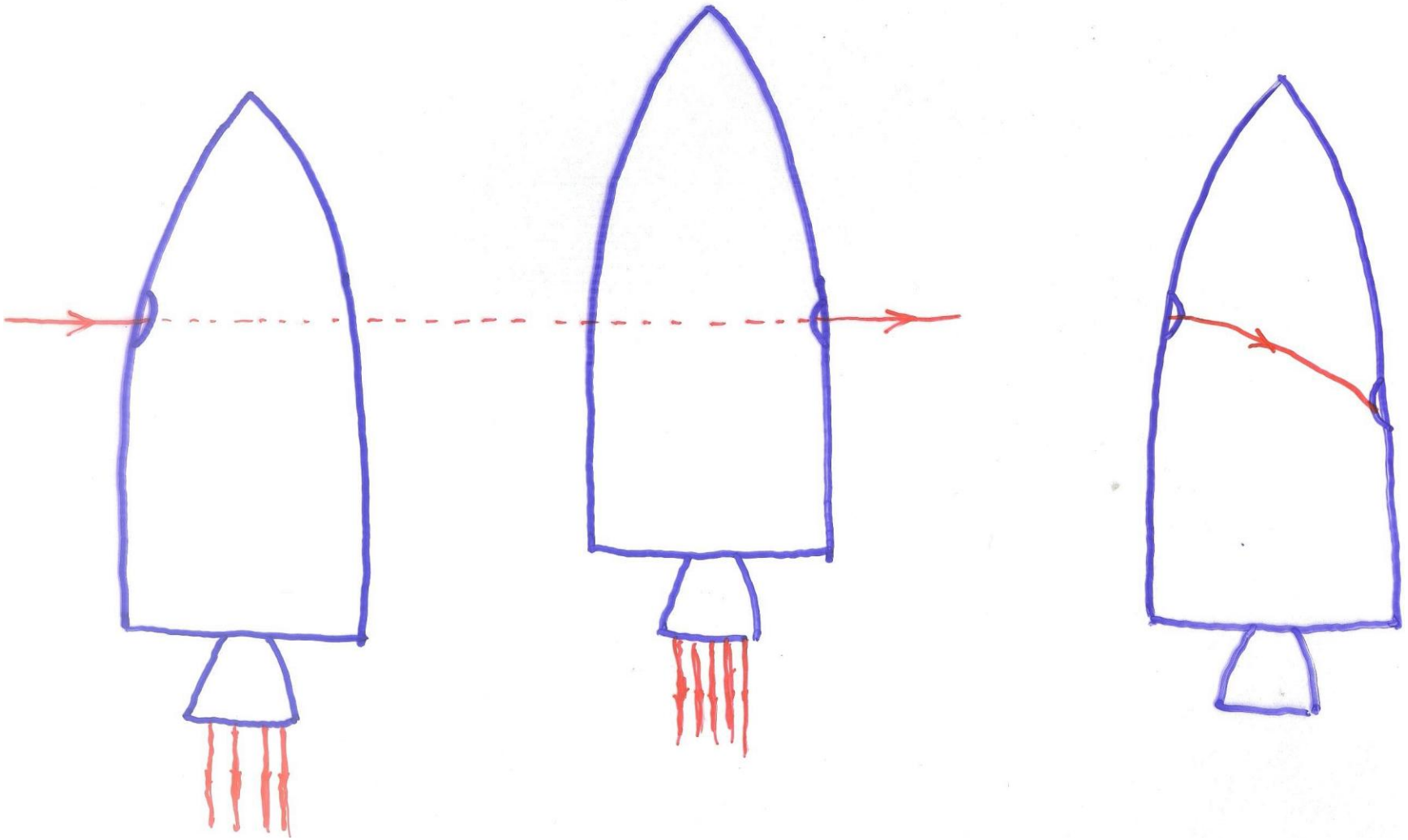
— 愛因斯坦
1907年
於伯恩專利局辦公室

“重力場和因為電磁感應而產生的電場類似，其存在只有相對的意義。因為對於一位從屋頂自由落下的觀測者而言—至少在其附近—重力場**並不存在**。”

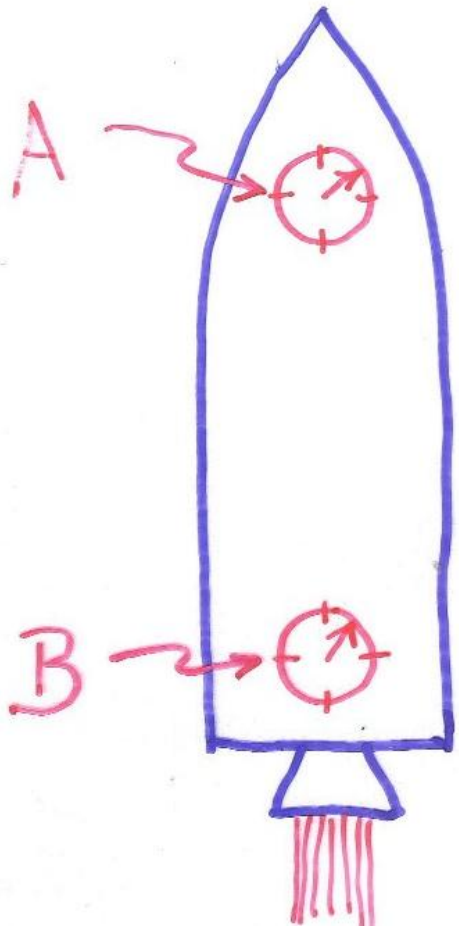


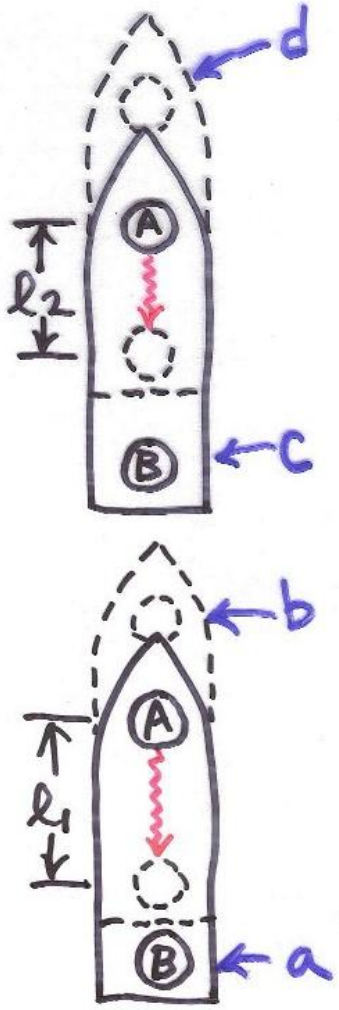
等效原理

Equivalence Principle



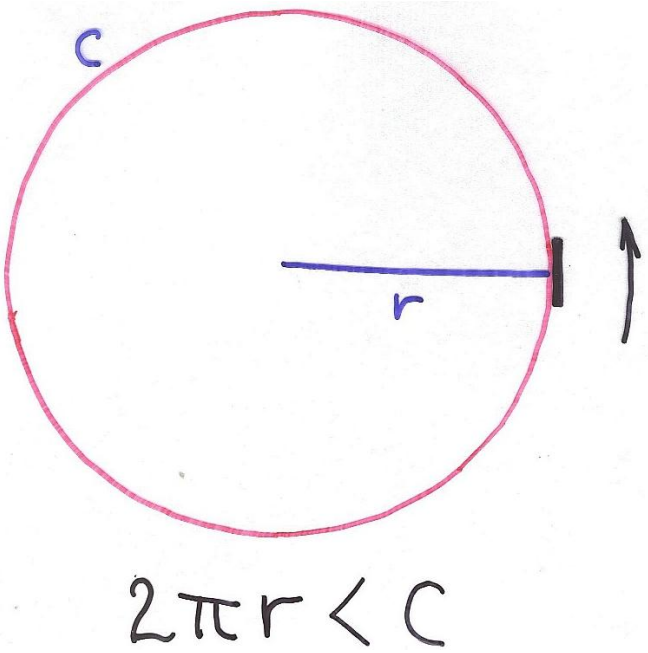
光線在重力場中會轉彎





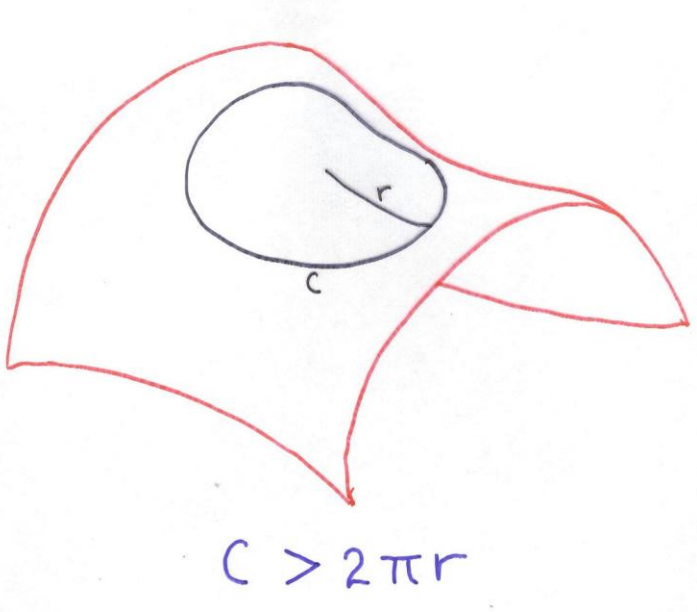
費曼物理學講義
第二冊

A鐘比B鐘走得快，依據等效原理，台北101**頂樓**的鐘也比**底樓**的鐘走得**快**。



由於長度收縮效應，
旋轉座標中的觀測者發現
圓周長c與半徑r的關係和
歐氏平面幾何中的關係不一樣。

→ **非歐幾何**



彎曲空間

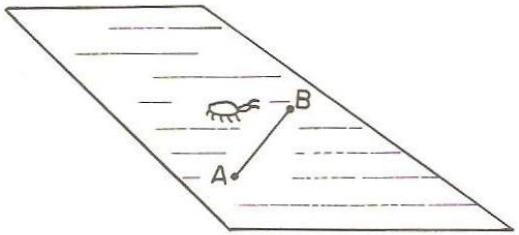


Fig. 42–4. Making a "straight line" on a plane.

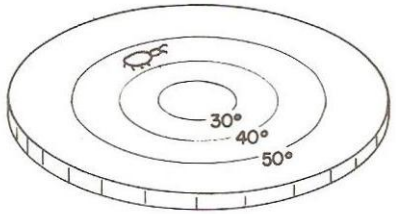


Fig. 42–3. A bug on a hot plate.

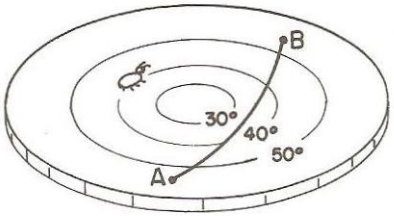


Fig. 42–6. Making a "straight line" on the hot plate.

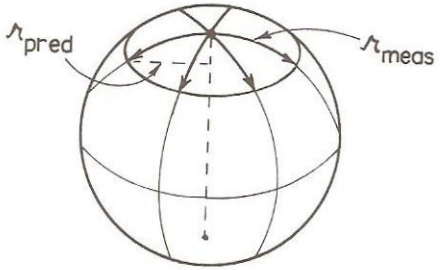


Fig. 42–11. Making a circle on a sphere.

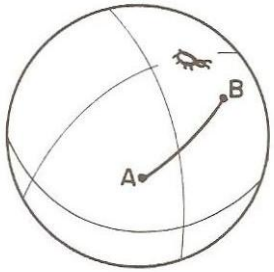
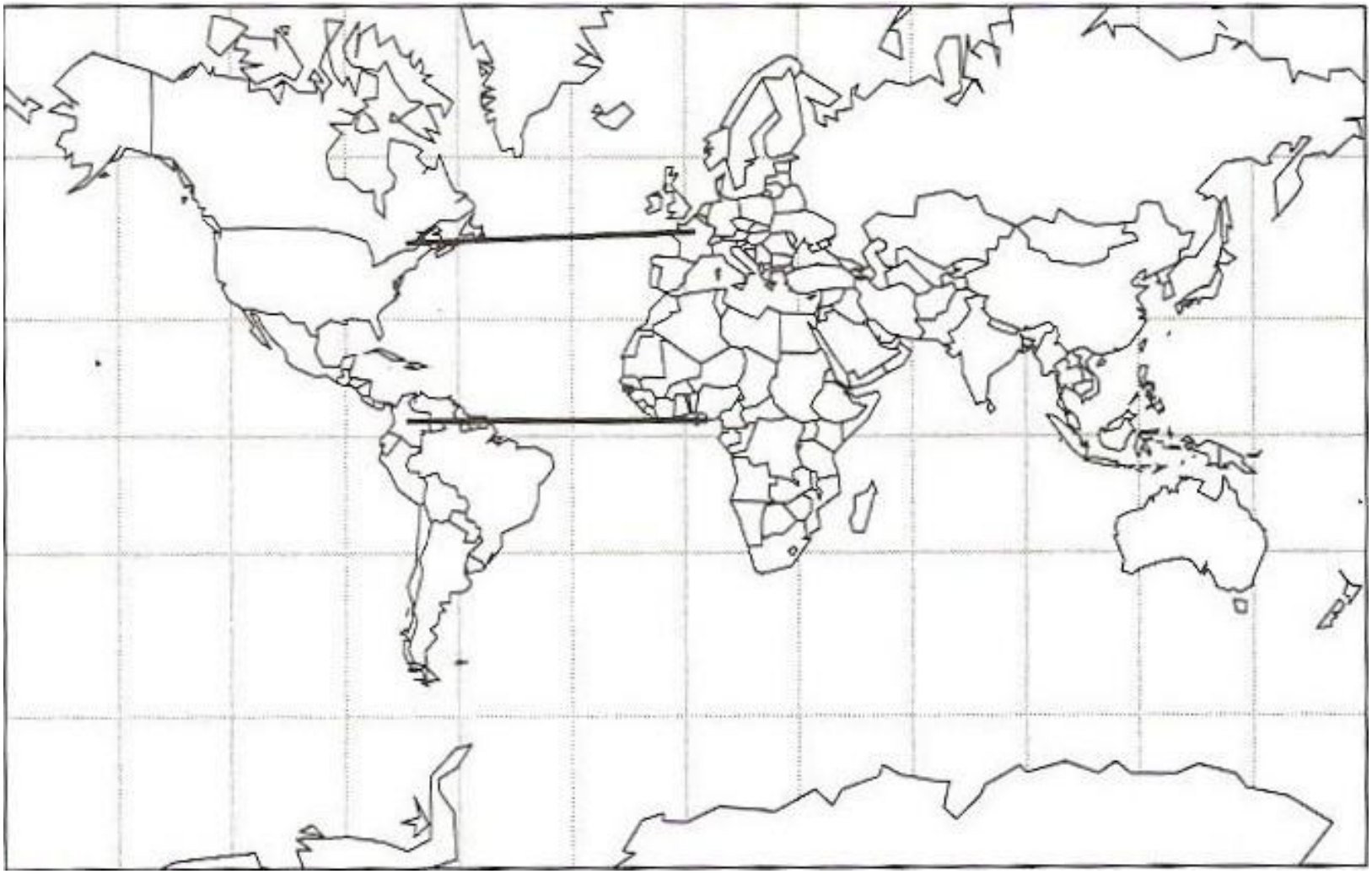


Fig. 42–5. Making a "straight line" on a sphere.

Feynman Lectures on Physics
Vol.2



Montreal-Paris < Bogota-Lagos ???

Flat-Earth Theory

J. Hartle *Gravity*

依據等效原理，
加速座標中的效應(現象)也會出現於
重力場中，
因此重力場中各處時鐘的快慢不一
而尺的長短也會不一樣。

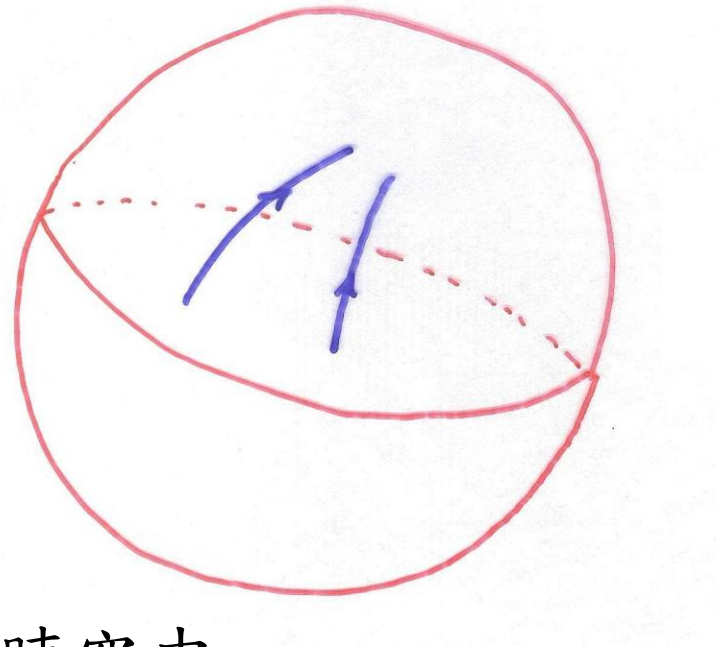
我們可以將重力場造成的效應看成是**彎曲時空**的效應！

傳統上，我們說物質會產生重力場，
在廣義相對論中我們說物質會改變其附近時空的性質—物質讓時空成了彎曲時空。

潮汐力 (tidal force)

重力 ≈ 幾何

Gravity ≈ Geometry



物體在重力場中的軌跡就是彎曲時空中的**極端線**(最短「距離」線)。

舞台

↑↓

時空

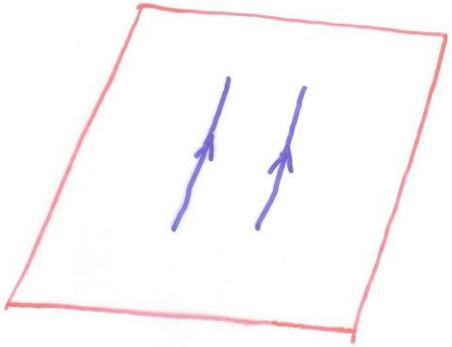
↑↓

相互影響

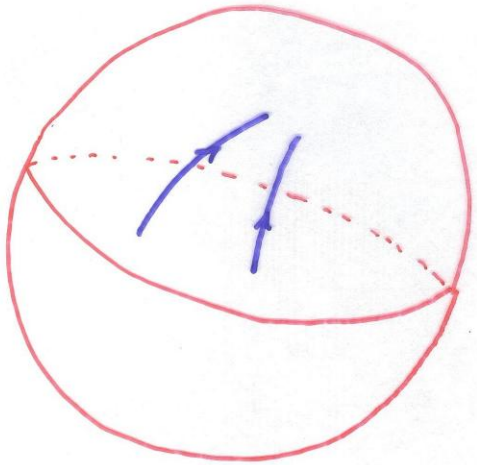
演員

物質

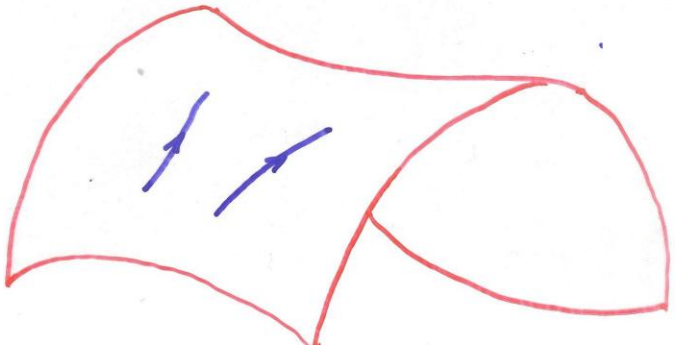
二維彎曲空間的例子



曲率＝0



曲率 ＞ 0



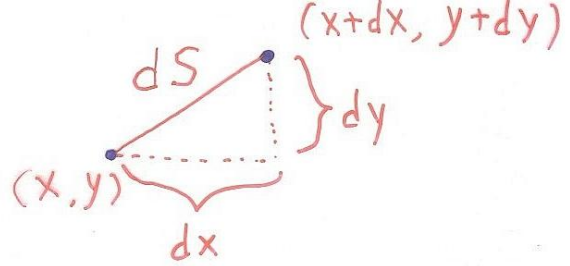
曲率 ＜ 0

如何描述彎曲空間的幾何？

黎曼Riemann
(1826-1866)

我們只需講明
空間中任何鄰近
兩點的**距離**

黎曼微分幾何

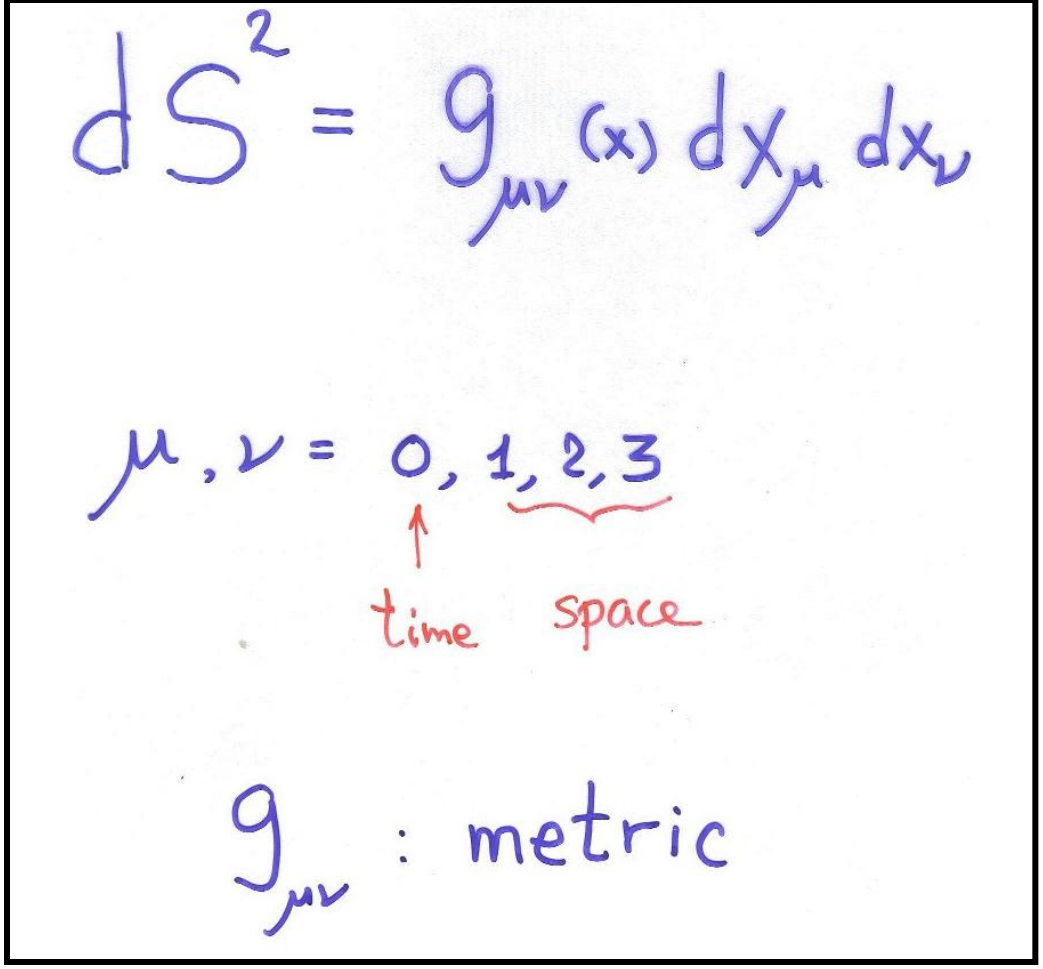


$$ds^2 = dx^2 + dy^2$$
$$= dx_1^2 + dx_2^2$$
$$= dx_1 \cdot dx_1 + dx_2 \cdot dx_2$$
$$= \sum_{i=1}^{2} dx_i \cdot dx_i$$
$$= \sum_{i=1}^{2} \sum_{j=1}^{2} g_{ij}\, dx_i\, dx_j$$

$$g_{11} = 1, \quad g_{12} = g_{21} = 0, \quad g_{22} = 1$$

$$\boxed{ds^2 = g_{ij}\, dx_i\, dx_j}$$

Einstein convention



度規(metric)決定彎曲

空間(時空)的性質(曲率)

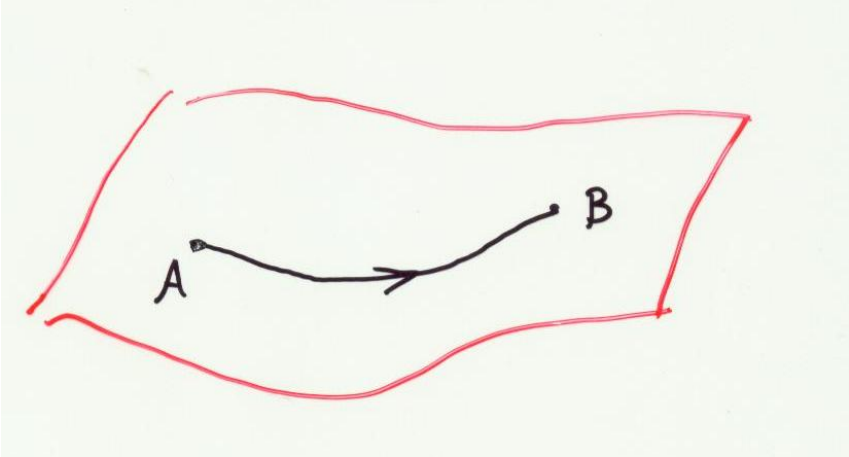
物質決定時空的曲率

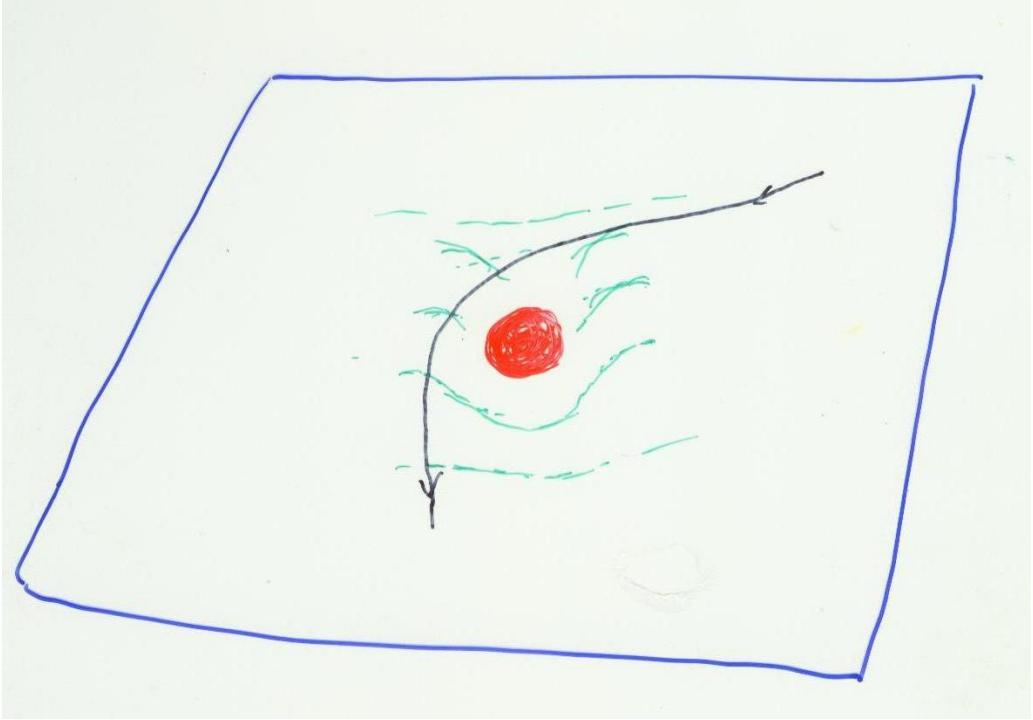
$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = k T_{\mu\nu}$$

愛因斯坦重力方程式(1915)

時空曲率決定物質(粒子)如何運動

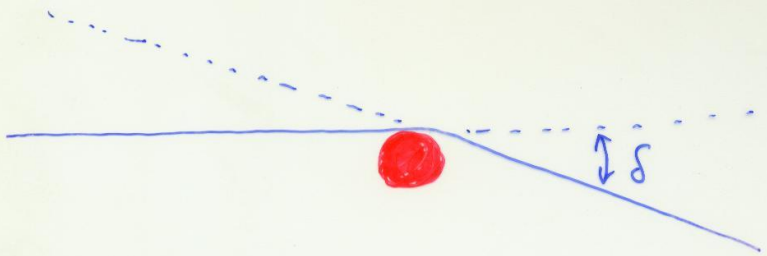
→ 粒子沿著極端線前進





物質(能量、動量)影響(改變)了時空曲率

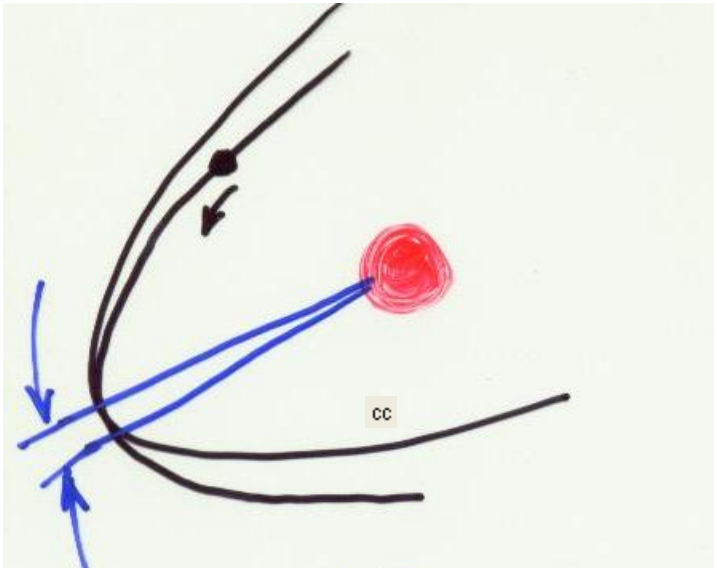
在時空中運動的粒子其軌跡就不再必然是一直線 → 粒子受重力影響而有彎曲的軌跡



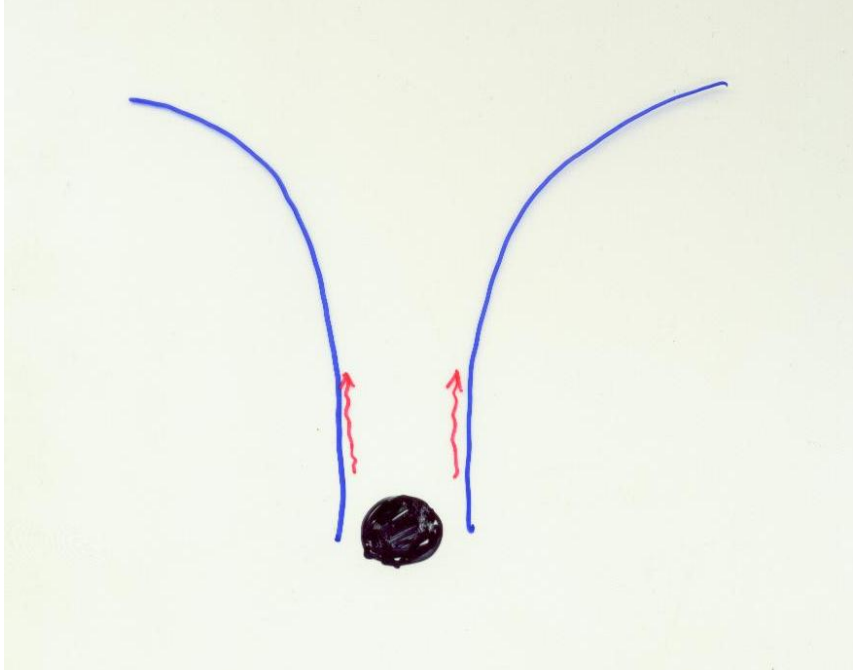
$\delta \approx 1.75''$

$(1^\circ = 3600'')$

水星近日點的進動(precession)



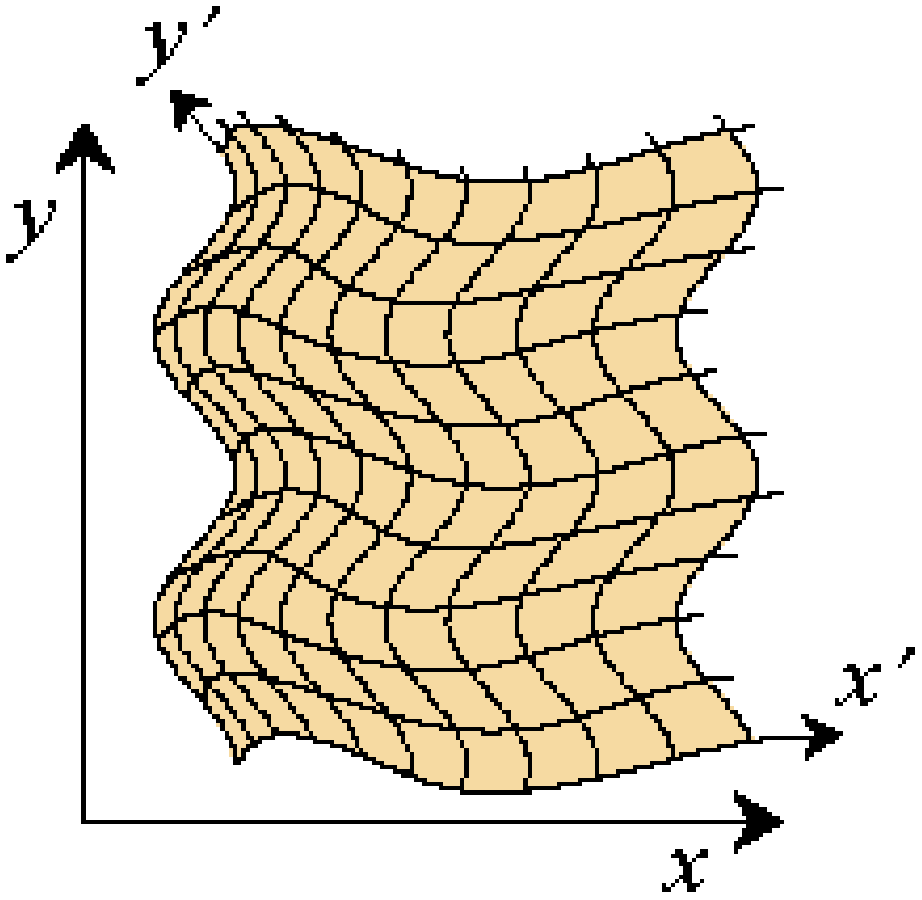
- 在~5600弧秒/世紀的進動之中有~43弧秒/世紀是來自廣義相對論效應。
- 大部分的進動來自於水星與其它行星的重力交互作用。



黑洞　Black Hole

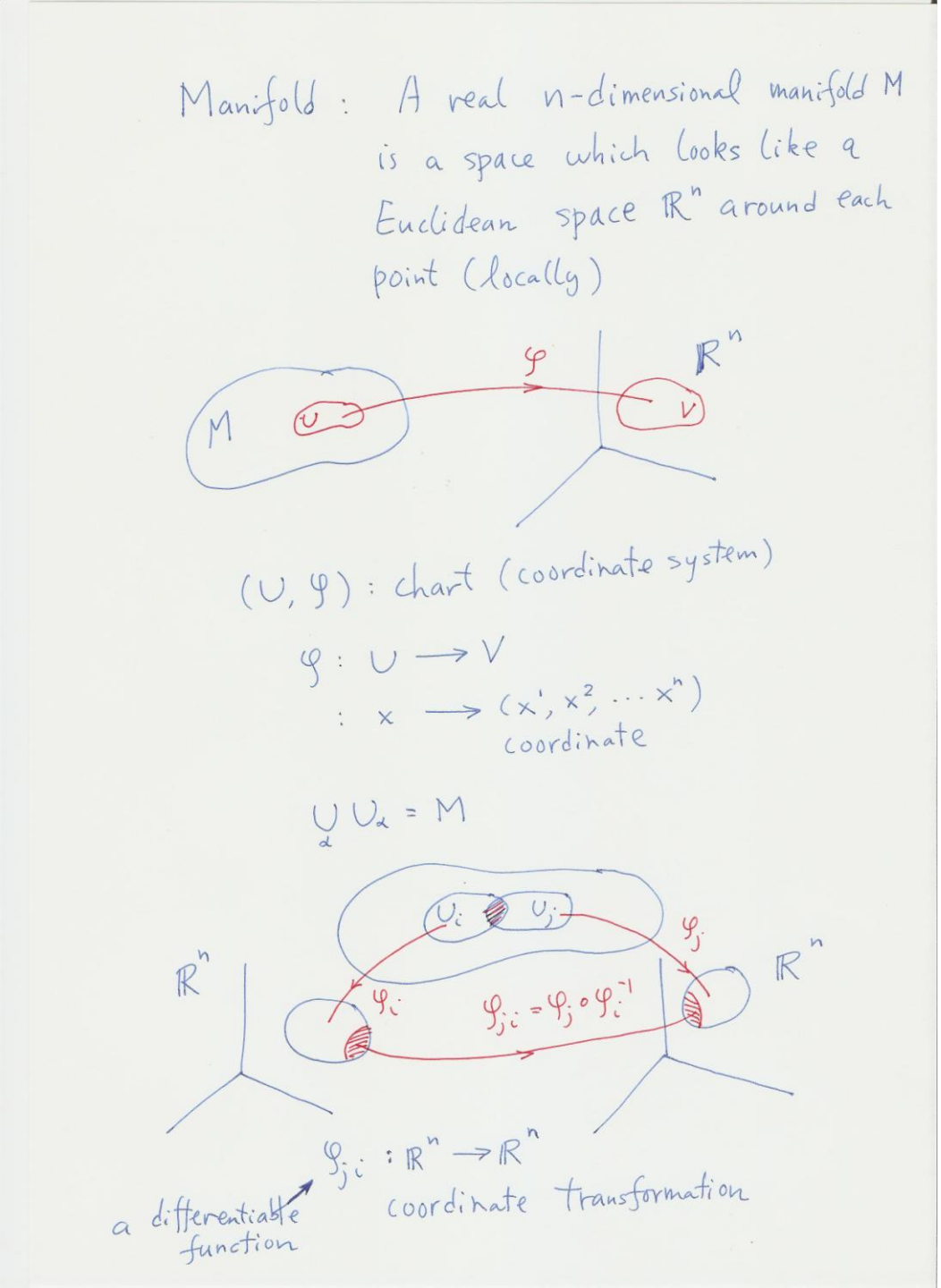
曲率大到沒有東西可以逃離

(曲率大到沒有極端線能連到無窮遠)



$$x^{\mu} \longrightarrow x^{\lambda}$$

The points of an *n*-dimensional manifold can thereby be labeled locally by coordinates $(x^1, , x^n)$, but these coordinate labels are arbitrary and could equally well be replaced by any other coordinate labels $(x^{\prime 1}, , x^{\prime n})$ that are related to $(x^1, , x^n)$ in a smooth, nonsingular manner



$$\phi'(u(x)) = \phi(x)$$

Scalar Field

Define a tangent vector at a point to be a collection of *n* numbers associated with a coordinate system that transforms by the vector-transformation law under a change of coordinates

However, it is not particularly intuitive. Furthermore, by tying the notion of a tangent vector to the presence of a coordinate system, it makes it extremely difficult for students to think about tangent vectors (and tensors) in a geometrical, coordinate-independent way.

Vectors and Tensors

Change of Coordinates

$$x^\mu \to x'^\mu = f^\mu(x^1, x^2, \cdots x^n) \qquad \mu = 1, 2, \cdots n$$

$$x'^\mu = x'^\mu(x)$$

A (contravariant) vector V^μ is a set of quantities that transform according to

$$V'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu(x)$$

A covariant vector ω_μ:

$$\omega'_\mu(x') = \frac{\partial x^\nu}{\partial x'^\mu} \omega_\nu(x)$$

A (contravariant) tensor of rank 2 $T^{\mu\nu}$:

$$T'^{\mu\nu}(x') = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} T^{\alpha\beta}(x)$$

A covariant tensor of rank 2 $T_{\mu\nu}$:

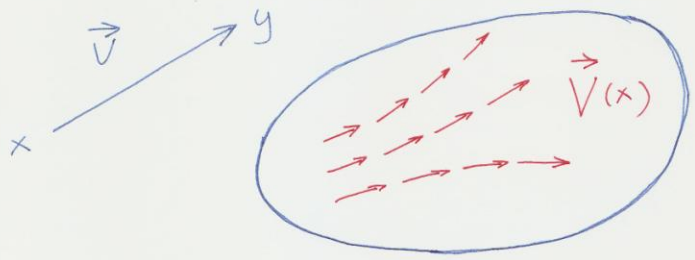
$$T'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} T_{\alpha\beta}(x)$$

"index-free" notation

Geometric Object:

Coordinate independent object

index-free interpretation of vector



$\vec{V} \leftrightarrow$ directional derivative

$\vec{V} \leftrightarrow V^\mu \frac{\partial}{\partial x^\mu} = V^\mu \partial_\mu$

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu},$$

$$\boxed{\partial'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu}$$

Since $\vec{V}$ is a geometric object,

$$\vec{V} = V'^{\mu} \partial'_{\mu} = V^{\mu} \partial_{\mu}$$

$$= V'^{\mu} \frac{\partial x^{\nu}}{\partial x'^{\mu}} \partial_{\nu}$$

$$V^{\nu}$$

$$\boxed{V^{\nu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} V'^{\mu}}$$

i.e.

$$\boxed{V'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}}$$

“However, very few students have any inkling that, in nature, the **points of space and the events in space-time fail to have any natural vector-space structure**. Indeed, the concept of a "vector" is normally introduced to students early in their physics education through the concept of "**position vectors**" representing the points of space. Students are taught that, given the choice of a point to serve as an "origin," it makes sense to add and scalar multiply points of space. ……**This situation changes dramatically in general relativity**, since the vector-space character of space and space-time depends crucially on having a **flat geometry**. In general relativity, it does not make any more sense to "add" two events in space-time than it would make sense to try to define a notion of addition of points on the surface of a potato. “

"4-vector" x^{μ} **? Yes in SR, No in GR**

R. B. Wald

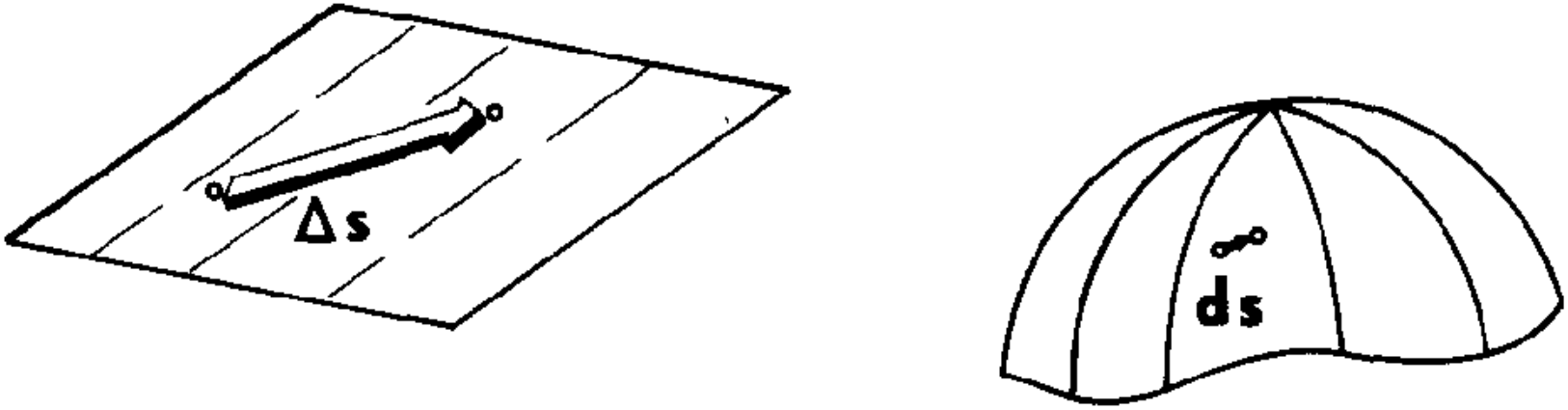


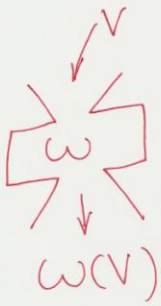
Fig. 4. Finite displacement as a vector in a two-dimensional flat space, and an infinitesimal displacement as a vector in a two-dimensional curved space.

At a given point, in an N-dimensional vector spacetime differential displacement vectors form an N-dimensional vector space.

A differential displacement is too small to care that the geometry is curved.

1-form ω: a linear real-valued function of vectors.

$$\omega : V \to \mathbb{R}$$
$$: v \to \omega(v)$$



$$\omega(v) \equiv v(\omega) \equiv \langle \omega, v \rangle$$

covector (or covariant vector)

vector (or contravariant vector)

If

$$\omega = \omega_\alpha \, dx^\alpha$$

$$V = V^\beta \partial_\beta$$

then

$$\omega(V) = \omega_\alpha V^\alpha$$

Since $\omega_\alpha V^\alpha = \omega'_\beta V'^\beta$ ∵ coordinate independent

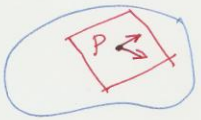
$$\therefore \quad \omega_\alpha \frac{\partial x^\alpha}{\partial x'^\beta} V'^\beta$$

$$\therefore \quad \boxed{\omega'_\beta = \frac{\partial x^\alpha}{\partial x'^\beta} \omega_\alpha}$$

i.e. $$\boxed{\omega'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} \omega_\nu}$$

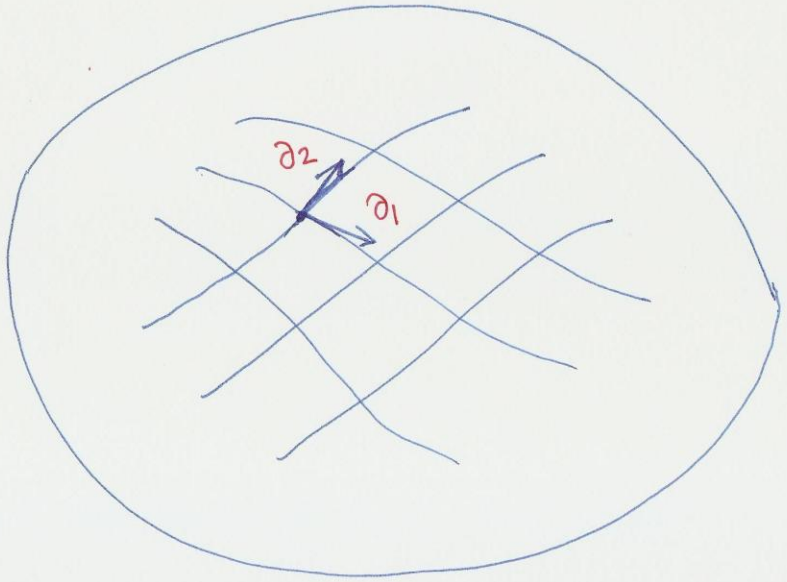
$$dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} dx^{\nu}$$

$$\left\{ \frac{\partial}{\partial x^{\mu}} \right\} = \left\{ \partial_{\mu} \right\}$$



defines a basis for tangent space (at point P) T_P

$\{ dx^{\mu} \}$ defines a dual basis for cotangent space T_P^*



$$\partial_\mu \longleftrightarrow \vec{e}_\mu$$

$$\partial_1 \longleftrightarrow \vec{e}_1$$

$$\partial_2 \longleftrightarrow \vec{e}_2$$

coordinate basis vector

A (k, ℓ) type tensor T:

$$T = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell} \, \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_k} \otimes dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_\ell}$$

$$T'^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell} = \frac{\partial x'^{\mu_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial x'^{\mu_k}}{\partial x^{\alpha_k}} \frac{\partial x^{\beta_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\beta_\ell}}{\partial x^{\nu_\ell}} \cdot T^{\alpha_1 \cdots \alpha_k}{}_{\beta_1 \cdots \beta_\ell}$$

A tensor of rank (k, ℓ) T

$$T : \underbrace{T_p^* \otimes \cdots \otimes T_p^*}_{k \text{ times}} \otimes \underbrace{T_p \otimes \cdots T_p}_{\ell \text{ times}} \to \mathbb{R}$$

Ex. $T = T_{\mu\nu}{}^{\alpha}\, \partial_\alpha \otimes dx^\mu \otimes dx^\nu$

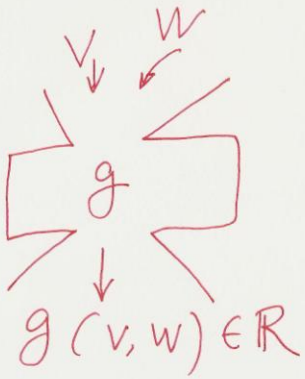
$$T(\omega, V, W)$$

$$= T_{\mu\nu}{}^{\alpha}\, \omega_\alpha V^\mu W^\nu$$

"index-free" notation for tensors

g: metric tensor

a $(0, 2)$ type tensor



$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$$

$g(V, W) = g_{\mu\nu} V^\mu W^\nu$: inner product of V, W

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = g'_{\mu\nu}\, dx'^\mu dx'^\nu$$

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$$

$$\tau = \int \left(-g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{1/2} d\lambda$$

the proper time functional

the proper time elapsed along the curve

$$\mathbf{A}\cdot\mathbf{B} = g_{\mu\nu}A^{\mu}B^{\nu}$$

$$\mathbf{A}\cdot\mathbf{B} = (A^{\mu}\mathbf{e}_{\mu})\cdot(B^{\mu}\mathbf{e}_{\nu}) = \mathbf{e}_{\mu}\cdot\mathbf{e}_{\nu}A^{\mu}B^{\nu}$$

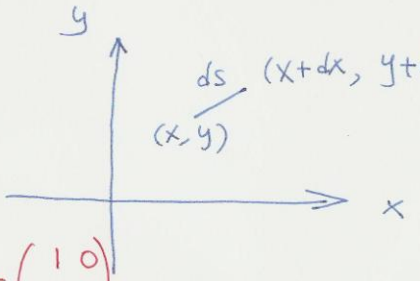
$$g_{\mu\nu} = \mathbf{e}_{\mu}\cdot\mathbf{e}_{\nu}$$

$$A_{\mu} \equiv g_{\mu\alpha}A^{\alpha}$$

$$g^{\beta\mu}g_{\mu\alpha} = \delta^{\beta}{}_{\alpha}$$

Examples of Metric Tensor

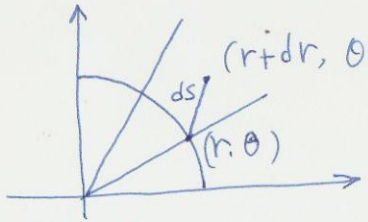
1. $x^1 = x$, $x^2 = y$



$$ds^2 = dx^2 + dy^2 = (dx^1)^2 + (dx^2)^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

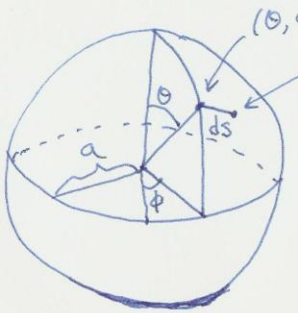
2. $x^1 = r$, $x^2 = \theta$



$$ds^2 = dr^2 + r^2 d\theta^2 = (dx^1)^2 + x^{1\,2}(dx^2)^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & x^{1\,2} \end{pmatrix}$$

3. S^2



$x^1 = \theta$, $x^2 = \phi$

$$ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$= a^2 (dx^1)^2 + a^2 \sin^2 x_1 (dx^2)^2$$

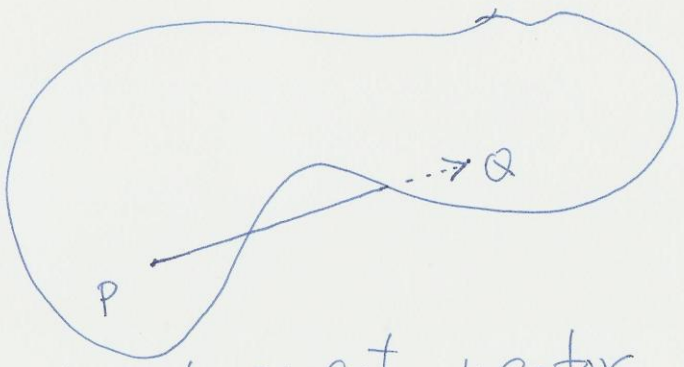
$$g_{\mu\nu} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \sin^2 x_1 \end{pmatrix}$$

Minkowski spacetime

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

$$\mathbf{ds} = dx^\mu \mathbf{e}_\mu \qquad \mathbf{ds}\cdot\mathbf{ds} = (ds)^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$



"Finite" displacement vector has no intrinsic geometric meaning (does not lie in the manifold)

Yet, we can define the displacement vector $d\vec{x}$ between 2 nearby points P and Q, since this is a local quantity. In the limit $Q \to P$, the vector $d\vec{x}$ lies in the tangent space at P.

“How does one go about giving a precise mathematical description of the geometry of a space-time in general relativity—or, for that matter, of the geometry of a surface of a potato? The notion of a "distance function" between (finitely separated) points can be defined for the surface of a potato, and, similarly, the notion of a "space-time interval" could be defined for (finitely separated, but sufficiently close) events in general relativity, but it would be **extremely cumbersome** to base a geometrical description of these entities on such a notion. A much better idea is to work **infinitesimally**, using the idea that, **on sufficiently small scales, a curved geometry looks very nearly flat**. These departures from flatness can then be described by differential calculus. To do so, one begins by introducing the notion of a tangent vector to describe an **infinitesimal displacement** about a point *p*.”

R. B Wald

Differentiation of a tensor does not generally produce another tensor

Ex. Consider a vector V^μ. $V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu$

$$\frac{\partial V'^\mu}{\partial x'^\lambda} = \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial V^\nu}{\partial x^\sigma}\frac{\partial x^\sigma}{\partial x'^\lambda} + \frac{\partial^2 x'^\mu}{\partial x^\sigma \partial x^\nu}\frac{\partial x^\sigma}{\partial x'^\lambda} V^\nu$$

$$= \frac{\partial x'^\mu}{\partial x^\nu}\frac{\partial x^\sigma}{\partial x'^\lambda}\frac{\partial V^\nu}{\partial x^\sigma} + \underbrace{\frac{\partial^2 x'^\mu}{\partial x^\sigma \partial x^\nu}\frac{\partial x^\sigma}{\partial x'^\lambda}}_{\text{in general} \neq 0} V^\nu$$

Covariant Derivative of V — connection coefficients

Define $\boxed{\nabla_\lambda V^\mu \equiv \partial_\lambda V^\mu + \Gamma^\mu_{\lambda\kappa} V^\kappa} = \frac{\partial V^\mu}{\partial x^\lambda} + \Gamma^\mu_{\lambda\kappa} V^\kappa$

with $\Gamma'^{\lambda}_{\mu\nu} \equiv \frac{\partial x'^\lambda}{\partial x^\sigma}\frac{\partial x^\tau}{\partial x'^\mu}\frac{\partial x^\rho}{\partial x'^\nu}\Gamma^\sigma_{\tau\rho} - \frac{\partial x^\sigma}{\partial x'^\nu}\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial^2 x'^\lambda}{\partial x^\sigma \partial x^\rho}$

Not a tensor!!!

So that $\nabla_\lambda V^\mu$ is a tensor $V^\mu{}_{;\lambda}$
(1,1) type

Because $$\boxed{\nabla'_\mu V'^\nu = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^\beta}\nabla_\alpha V^\beta}$$

Proof: straightforward calculation

Similarly $$\boxed{\nabla_\mu \omega_\nu \equiv \partial_\mu \omega_\nu - \Gamma^\lambda{}_{\mu\nu}\omega_\lambda}$$

: covariant derivative of a 1-form

Although $\Gamma^\lambda{}_{\mu\nu}$ is not a tensor,

$\underbrace{T^\lambda{}_{\mu\nu}}_{\text{torsion tensor}} \equiv \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}$ is a tensor

$$\{\partial_\alpha\} = \{e_\alpha\}$$

$$V = V^\beta e_\beta$$

$$\nabla_\alpha V = \nabla_\alpha (V^\beta e_\beta)$$

$$= (\nabla_\alpha V^\beta)\, e_\beta + V^\beta (\nabla_\alpha e_\beta)$$

$$= (\partial_\alpha V^\beta + \Gamma^\beta_{\alpha\mu} V^\mu) e_\beta$$

$\nabla_\alpha e_\beta = \Gamma^\mu_{\alpha\beta}\, e_\mu$

affine connection or Christoffel symbol

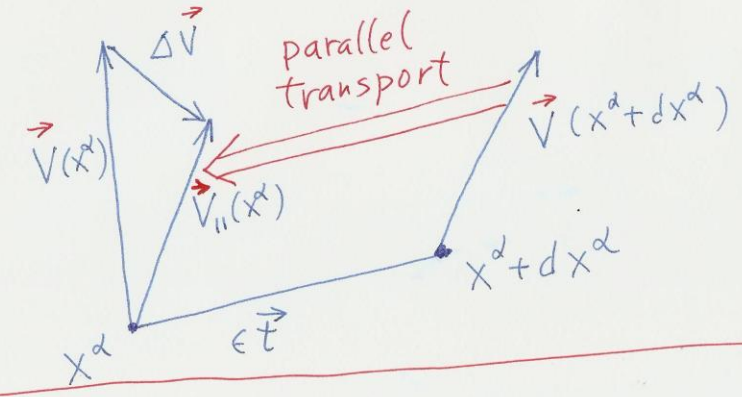
$\partial_\alpha \vec{V} = \partial_\alpha (V^\beta \vec{e}_\beta)$

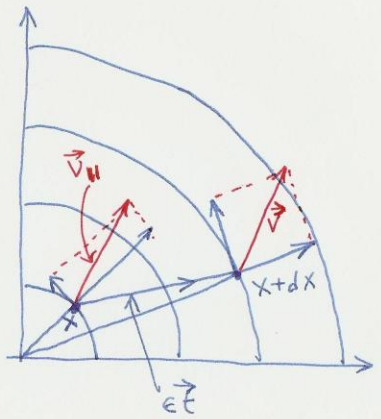
$= (\partial_\alpha V^\beta)\, \vec{e}_\beta + V^\beta (\partial_\alpha \vec{e}_\beta)$

$= (\partial_\alpha V^\beta + \Gamma^\beta_{\alpha\mu} V^\mu)\, \vec{e}_\beta$

$\partial_\alpha \vec{e}_\beta \equiv \Gamma^\mu_{\alpha\beta}\, \vec{e}_\mu$

Geometric Picture :





See Hartle Gravity

$$\nabla_{\vec{t}}\,\vec{V}(x^\alpha) \equiv \lim_{\epsilon\to 0} \frac{\vec{V}(x^\alpha+\epsilon t^\alpha)_{\parallel \text{ trans to } x^\alpha} - \vec{V}(x^\alpha)}{\epsilon}$$

$$V^\alpha_{\parallel}(x) = V^\alpha(x+\epsilon t) + \Gamma^\alpha_{\beta\gamma}(x)\,V^\gamma(x)\cdot\epsilon\, t^\beta$$

from the change in the basis vectors

$$\therefore \quad \nabla_\mu V^\nu = \frac{\partial V^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu\alpha}\,V^\alpha$$

$$\nabla : \underset{(k,\ell)}{T} \longrightarrow \underset{(k,\ell+1)}{T'}$$

$$\nabla_\sigma T^{\mu_1 \mu_2 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell} = \partial_\sigma T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}$$

$$+ \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda \mu_2 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1 \lambda \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}$$

$$+ \cdots\cdots$$

$$- \Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1 \cdots \mu_k}{}_{\lambda \nu_2 \cdots \nu_\ell} - \Gamma^{\lambda}_{\sigma\nu_2} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \lambda \cdots \nu_\ell}$$

$$- \cdots\cdots\cdots$$

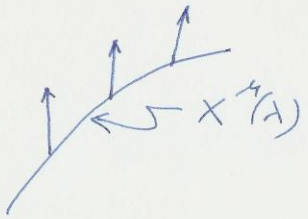
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2 properties:

1. Linearity : $\nabla(T+S) = \nabla T + \nabla S$
2. Leibniz rule : $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$

Directional derivative along a curve $X^\mu(\lambda)$

For flat space, $\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{\partial}{\partial x^\mu}$



directional covariant derivative:

$$\boxed{\frac{D}{d\lambda} \equiv \frac{dx^\mu}{d\lambda}\nabla_\mu}$$

If $\boxed{\frac{D}{d\lambda}T = 0}$, tensor T is parallel transported along curve $X^\mu(\lambda)$.

Ex. for a vector V

$$\frac{DV}{d\lambda} = 0 \rightarrow \boxed{\frac{d}{d\lambda}V^\mu + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}V^\rho = 0}$$

equation of parallel transport

definition: A connection Γ is **metric-compatible** if the metric is always parallel transported with respect to it, i.e.

$$\frac{D}{d\lambda} g_{\mu\nu} = \frac{dx^\sigma}{d\lambda} \nabla_\sigma g_{\mu\nu} = 0 \quad \left(\text{or } \nabla_\sigma g_{\mu\nu} = 0 \right)$$

If $\nabla_\sigma g_{\mu\nu} = 0$, the inner product of 2 parallel-transported vectors is preserved:

$$\frac{D}{d\lambda}\left(g_{\mu\nu} V^\mu W^\nu\right) = \left(\frac{D}{d\lambda} g_{\mu\nu}\right) V^\mu W^\nu + g_{\mu\nu}\left(\underbrace{\frac{D}{d\lambda} V^\mu}_{=0}\right) W^\nu$$

$$+ g_{\mu\nu} V^\mu \left(\underbrace{\frac{D}{d\lambda} W^\nu}_{=0}\right) \quad \because V, W \text{ are parallel transported}$$

$$= 0$$

If Γ is torsion-free & metric compatible, i.e.,

1) $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$

& 2) $\nabla_\alpha g_{\mu\nu} = 0$ then

$$\boxed{\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)} = \left\{ {\sigma \atop \alpha\beta} \right\}$$

Proof:

$$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^{\lambda}_{\rho\mu} g_{\lambda\nu} - \Gamma^{\lambda}_{\rho\nu} g_{\mu\lambda} = 0$$

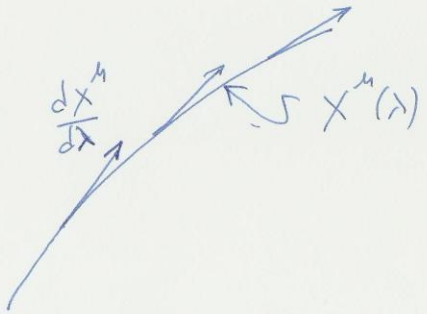
$$+) \quad \nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho} - \Gamma^{\lambda}_{\mu\rho} g_{\nu\lambda} = 0$$

$$-) \quad \nabla_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - \Gamma^{\lambda}_{\nu\rho} g_{\lambda\mu} - \Gamma^{\lambda}_{\nu\mu} g_{\rho\lambda} = 0$$

$$g^{\sigma\rho} \times \left(\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2\Gamma^{\lambda}_{\mu\nu} g_{\lambda\rho}\right) = 0$$

$$\hookrightarrow \Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)$$

Geodesic : a curve along which the tangent vector is parallel-transported.



$$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = 0$$

$$\therefore \quad \boxed{\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0}$$

the geodesic equation

$$S \equiv \int_C \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} = \text{length of the curve } C$$

$\delta S = 0$ → same geodesic equation

extremal length

Riemann curvature tensor $R^{\alpha}{}_{\beta\mu\nu}$:

$$[\nabla_\mu, \nabla_\nu] \equiv \nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu$$

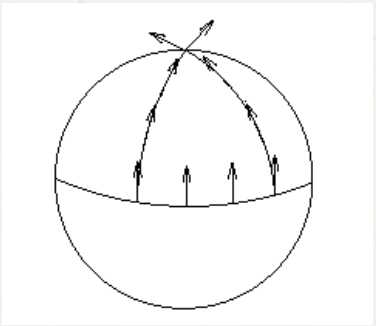
$$[\nabla_\mu, \nabla_\nu] V^\alpha = R^\alpha{}_{\beta\mu\nu} V^\beta - T^\lambda_{\mu\nu} \nabla_\lambda V^\beta$$

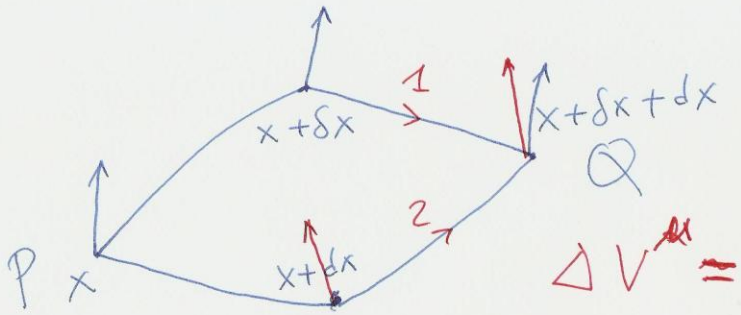
$$\boxed{R^\alpha{}_{\beta\mu\nu} \equiv \partial_\mu \Gamma^\alpha{}_{\nu\beta} - \partial_\nu \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\nu\beta} - \Gamma^\alpha_{\nu\lambda} \Gamma^\lambda_{\mu\beta}}$$

$$R^{\cdot}{}_{\cdot\mu\nu} = \partial_\mu \Gamma^{\cdot}_{\nu\cdot} + \partial_\nu \Gamma^{\cdot}_{\mu\cdot} + \Gamma^{\cdot}_{\mu\lambda} \Gamma^{\lambda}_{\nu\cdot} - \Gamma^{\cdot}_{\nu\lambda} \Gamma^{\lambda}_{\mu\cdot}$$

definite : An affine flat manifold is a manifold with $\Gamma^{\mu}_{\alpha\beta} = 0$ everywhere

Theorem : A manifold is affine flat $\Longleftrightarrow R^{\alpha}_{\beta\mu\nu} = 0$





$$\Delta V^{\mu} = R^{\mu}{}_{\alpha\eta\beta} V^{\alpha} dx^{\eta} \delta x^{\beta}$$

definition : Ricci tensor $R_{\alpha\beta}$

$$R_{\mu\nu} \equiv R^{\alpha}_{\ \mu\alpha\nu}$$

definition : Curvature scalar R

$$R \equiv g^{\mu\nu} R_{\mu\nu}$$

Einstein's equation :

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$T_{\mu\nu}$: the energy-stress tensor of matter

$$R_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu})$$

geodesic equation $\longrightarrow$ Newton's eq of motion

$\Big\downarrow$ for static field & slow motion

$(c=1)$

$g_{00} = -1 - 2\phi$

mass density

$\vec{\nabla}^2\phi = 4\pi G\rho$

$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$

= Einstein tensor

$-\vec{\nabla}^2 g_{00} = 8\pi G T_{00}$

$G_{00} \cong -\vec{\nabla}^2 g_{00}$

$G_{00} = 8\pi G T_{00}$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\nabla^\mu G_{\mu\nu} = 0, \quad \nabla^\mu T_{\mu\nu} = 0$$

Bianchi identity

the Hilbert action

$$S_H = \int d^n x \mathcal{L}_H \qquad \mathcal{L}_H = \sqrt{-g} R$$

$$\delta S = \int d^n x \sqrt{-g} \left[R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right] \delta g^{\mu\nu}$$

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0$$

$$S = \frac{1}{8\pi G} S_H + S_M$$

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{8\pi G} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} = 0$$

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}$$

Cosmological Constant Λ:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G T^{\mu}_{\mu\nu}$$

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \left(T^{\mu}_{\mu\nu} - \rho_{vac} g_{\mu\nu} \right)}$$

$\rho_{vac} \equiv \dfrac{\Lambda}{8\pi G}$: vacuum energy density

Schwarzschild Solution

Ansatz: $$ds^2 = -A(r)\,dt^2 + B(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

static, isotropic metric

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G\, T_{\mu\nu} \propto M\delta(r)$$

$r \neq 0$ $\quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \rightarrow R_{\mu\nu} = 0$

$$g_{\mu\nu} \longrightarrow \Gamma^{\alpha}_{\mu\nu} \longrightarrow R^{\alpha}{}_{\mu\nu\beta} \longrightarrow R_{\mu\nu} \longrightarrow R_{\mu\nu} = 0$$

$g_{\mu\nu}$ $(A(r), B(r))$

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{1}{1 - \frac{2GM}{r}} dr^2 + r^2 d\Omega^2$$

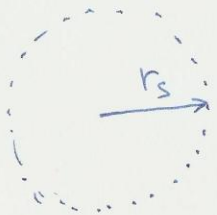
$$r_s \equiv \frac{2GM}{c^2} = \text{Schwarzschild radius}$$

$$g_{rr} \xrightarrow{r \to r_s} \infty \; ???$$

$\rightarrow$ coordinate singularity

But $g \xrightarrow{r \to 0} \infty$

$\rightarrow$ true singularity $\left(R^{\mu\nu\sigma\rho} R_{\mu\nu\sigma\rho} = \frac{48 G^2 M^2}{r^6} \right)$



Event horizon

Black Hole

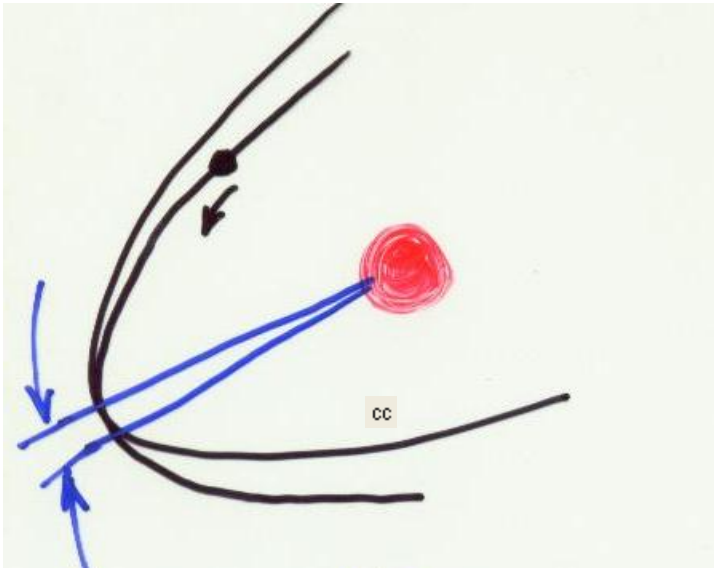
Geodesics of Schwarzschid spacetime:

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r)$$

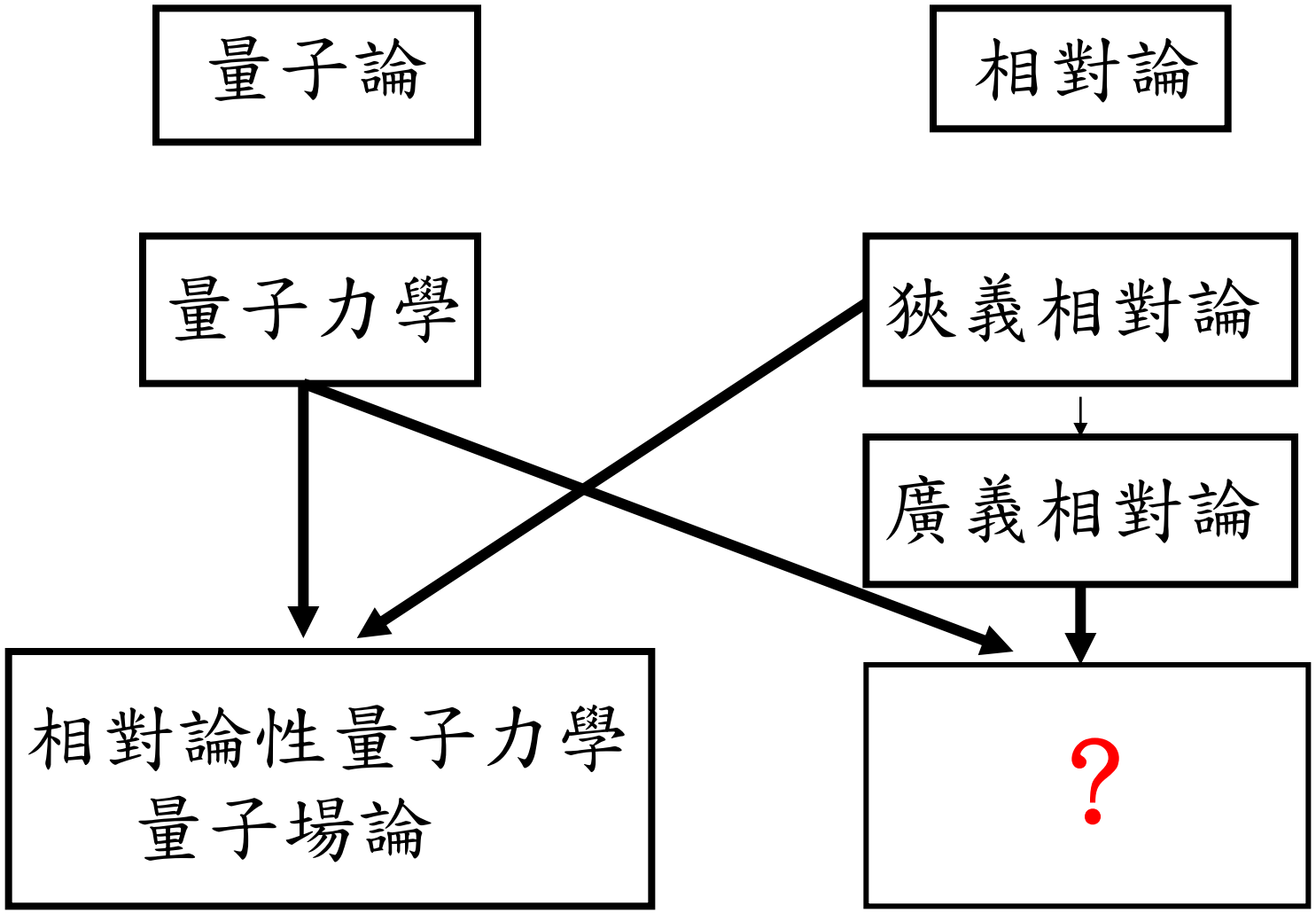
$$V_{eff}(r) = \frac{1}{c^2}\left(-\frac{GM}{r} + \frac{l^2}{2r^2} - \underbrace{\frac{GMl^2}{c^2 r^3}}\right)$$

leading to the precession of perihelion of planets

水星近日點的進動(precession)



- 在~5600弧秒/世紀的進動之中有~43弧秒/世紀是來自廣義相對論效應。
- 大部分的進動來自於水星與其它行星的重力交互作用。



能夠精準地描述基本粒子現象與凝體現象，是非常成功的理論。